Design-based causal inference in bipartite experiments

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Abstract

Bipartite experiments are widely used across various fields, yet existing methods often rely on strong assumptions about the potential outcomes modeling and exposure mapping. In this paper, we explore design-based causal inference in bipartite experiments. We first formulate the causal inference problem under a design-based framework that generalizes the classic assumption to account for bipartite interference. We then propose point and variance estimators for the total treatment effect, establishing a central limit theorem for the estimator and proposing a conservative variance estimator to ensure asymptotically valid inference. Additionally, we discuss a covariate adjustment strategy to enhance efficiency. We also conduct simulation and empirical studies to evaluate the finite-sample performance of the proposed estimators and validate our theoretical results.

Key Words: bipartite interference; finite population; covariate adjustment; randomization inference

1 Introduction

Bipartite experiments have gained increasing recognition for their utility in various fields, including digital experimentation, environmental science, and education. In bipartite experiments, treatments are randomized over one set of units, called *treatment units*, while outcomes are measured on a separate set of units, called *outcome units*. This is different from the classical experiment settings where we randomly assign units to different treatment groups and measure their outcome of interest after the initiation of treatment. As illustrated in Figure 1, the treatment units and outcome units are connected through a known fixed bipartite

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graph, and causal dependencies are represented by the bipartite network, leading to what is known as *bipartite interference* (Zigler and Papadogeorgou, 2021).

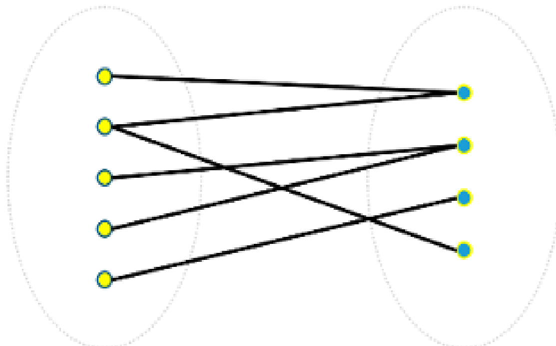

Figure 1: Illustration of a bipartite experiment

There has been extensive research on bipartite experiments. A well-known special case is cluster randomization, which has been studied in theory and adopted in practice (Donner, 1998; Donner et al., 2000; Su and Ding, 2021). The cluster experiment setup corresponds to a situation where each outcome unit is connected to exactly one intervention unit. More recent works have focused on the general bipartite network, where each outcome unit may be connected to multiple intervention units, and vice versa. From an analysis perspective, Zigler and Papadogeorgou (2021) formulated a set of causal estimands in bipartite experiments and proposed an inverse probability-weighted estimator for observational studies. Doudchenko et al. (2020) leveraged the generalized propensity score to obtain unbiased estimates of causal effects. Harshaw et al. (2023) explored the estimation and inference of the average total treatment effect under a linear exposure-response model. Shi et al. (2024) extended Harshaw et al. (2023) by studying covariate adjustment under the double linear model. Additionally, Song and Papadogeorgou (2024) studied bipartite experiments in the time series and random network setting under exposure examined bipartite experiments in time series and random network settings using exposure mapping and matching estimators for observational studies. Several works have also addressed the design of experiments in the presence of bipartite interference. For example, Pouget-Abadie et al. (2019), Harshaw et al. (2023), and Brennan et al. (2022) investigated methods for constructing better experimental designs in such context.

Despite these advances, there are gaps in the literature. Many existing approaches (e.g. Harshaw et al., 2023; Doudchenko et al., 2020) rely on strong structural modeling assumptions about potential outcomes, which makes valid inference heavily dependent on correct model specifications. For instance, Harshaw et al. (2023) adopted an exposure mapping (Aronow and Samii, 2017; Forastiere et al., 2021) with a linear outcome model, and Doudchenko et al. (2020) used exposure mapping for estimation along with a linear model for variance estimation. The performance of these model-based methods is sensitive to whether these assumptions align with the real data.

In contrast, a design-based perspective defines casual parameters based on potential outcomes and derives properties of regression estimators under treatment randomization (Neyman, 1990; Imbens and Rubin, 2015; Ding, 2024). Such a design-based perspective requires fewer assumptions and is generally more flexible. However, it has not yet been rigorously discussed in the context of bipartite experiments. There are several challenges to adopting a design-based framework in this context. For instance, how we can establish central limit theorems and construct valid variance estimators in the presence of strong dependencies across outcome units? Additionally, if covariate information is available, how should we perform covariate adjustment in a model-assisted fashion without relying on parametric models, following the spirit of Lin (2013)? These questions need to be addressed from a design-based perspective.

Our contributions. We contribute to filling the gaps in the literature with several key advancements. First, we formulate the causal inference problem in bipartite experiments using the design-based framework. We generalize the stable unit treatment value assumption (SUTVA) to account for bipartite interference. This generalization is tailored to the bipartite network structure, enabling the identification of the total treatment effect as a function of observed data. Unlike model-based frameworks, our approach avoids the need for assumptions regarding outcome modeling or exposure mapping.

Second, we propose a Hájek estimator for the total treatment effect and prove its consistency and asymptotic normality under certain assumptions on the network structure. We also propose a conservative variance estimator that ensures valid inference by accounting for the complexity of the network.

Third, we present a model-agnostic covariate-adjusted estimator that is asymptotically no less efficient than the naive approach while maintaining valid inference.

Organization of the paper. The rest of the paper is organized as follows. Section 2 introduces the design-based setup of the bipartite experiments. Section 3 discusses the identification and estimation of the total treatment effect under bipartite interference. Section 4 presents a covariate adjustment strategy for constructing point and variance estimators and proves their asymptotic properties. Section 5 conducts many numerical experiments to validate the proposed methods and theoretical results. Finally, Section 6 concludes the paper and outlines future research directions. All proofs and technical details are in the Supplementary Material.

Notation. We introduce the following notation. Let $\mathbb{1}\{\cdot\}$ denote the indicator function. Let plim denote the probability limit, avar$(\cdot)$ and acov$(\cdot,\cdot)$ denote the asymptotic variance and covariance, respectively, and $\asymp$ denote asymptotically the same order. For any positive integer K, denote $[K] = \{1, \ldots, K\}$ as the set of all positive integers smaller than or equal to K. Write $b_n = O(a_n)$ if b_n/a_n is bounded and $b_n = o(a_n)$ if b_n/a_n converges to 0 as $n \to \infty$. Also write $b_n = O_p(a_n)$ if b_n/a_n is bounded in probability and $b_n = o_p(a_n)$ if b_n/a_n converges to 0 in probability.

2 Setup

2.1 Motivating examples

We first present several motivating examples that highlight the applicability of bipartite experimental setups, which we will formalize in the next subsection through mathematical notation.

Example 1 (Power plant). *Zigler and Papadogeorgou (2021) illustrates a case to evaluate how the installation of selective noncatalytic reduction system or not (treatment) in their upwind power plants (treatment units) causally affects the hospitalization rates in the neighborhoods (outcome units). In this case, a neighborhood can be affected by the treatments of multiple upwind power plants, while one power plant may affect a set of neighborhoods. We will revisit this example in Section 5.2 as our real-world application.*

Example 2 (A/B testing and Identity fragmentation). *Shankar et al. (2023) introduces an online experiment where treatments are randomized across mobile devices and outcomes are measured from user-level units. This is a Bipartite Experiment as one user could have multiple devices to receive treatment, while one device may be shared by multiple users.*

Example 3 (Facebook Group). *Shi et al. (2024) considers a bipartite experiment where treatments are randomized across Facebook Groups and outcomes are measured by user-level engagement. The outcome of each user is affected by interventions on a set of groups she/he belongs to, while treatment in each group affects all users within that group.*

Example 4 (Amazon market). *Harshaw et al. (2023) simulates a bipartite experiment on the Amazon marketplace to evaluate the impact of new pricing mechanisms (treatments) randomized across items on customer satisfaction (outcome). In this scenario, items with new pricing mechanisms may influence the group of customers who view them, while each customer may encounter a variety of items subject to different pricing strategies.*

2.2 Setup of bipartite experiments

There are n units and m groups in the finite population. The treatment assignment is randomly assigned at the group level, while the outcome of interest is measured on the unit side. The bipartite network is summarized by a known $n \times m$ adjacency matrix $\boldsymbol{W}$, where W_{ik} equals 1 if unit i is in the group k, and 0 otherwise for $i = 1, \ldots, n$ and $k = 1, \ldots, m$. Let $\mathcal{S}_i \subset \{1, \ldots, m\}$ denote the subset which includes indices of all groups unit i is in, i.e., $k \in \mathcal{S}_i$ if and only if $W_{ik} = 1$, and let $|\mathcal{S}_i| = \sum_{k=1}^{m} W_{ik}$ denote the total number of groups unit i belongs to and $\bar{S} = \max_i |\mathcal{S}_i|$ denote the maximum number of groups the units belong to.

On the group side, we randomly assign the m groups to treatment and control arms. Let Z_k denote the binary treatment status of group k, where Z_k equals 1 if group k is assigned to treatment, and 0 otherwise. Let $\mathcal{D}_k \subset \{1, \ldots, n\}$ denote the subset which includes indices of all units that belong to group k, and $|\mathcal{D}_k| = \sum_{i=1}^{n} W_{ik}$ denote the number of units group k contains and $\bar{D} = \max_k |\mathcal{D}_k|$ denote the maximum number of units the groups contain.

For each unit i, there are 2^m potential outcomes $Y_i(\boldsymbol{z})$, where $\boldsymbol{z} = (z_1, \ldots, z_m)$ and $z_k = 0, 1$, $k = 1, \ldots, m$. The observed outcome $Y = Y(\boldsymbol{Z})$, where $\boldsymbol{Z} = (Z_1, \cdots, Z_m)$. We make the following assumption:

Assumption 1. *The potential outcomes of unit i depend only on the treatment status of the groups to which it belongs. Formally, $Y_i(\boldsymbol{z}) = Y_i(\boldsymbol{z}_{\mathcal{S}_i})$, where $\boldsymbol{z}_{\mathcal{S}_i}$ denotes the subvector of $\boldsymbol{z}$ corresponding to groups in $\mathcal{S}_i$.*

Here the potential outcome $Y_i(\boldsymbol{z})$ depends on the treatment vector $\boldsymbol{z}_{\mathcal{S}_i}$, whose dimension varies across units, unlike the classical setting. The causal parameter of interest is the total treatment effect

$$\tau = n^{-1} \sum_{i=1}^{n} \{Y_i(\mathbf{1}) - Y_i(\mathbf{0})\}.$$

τ captures the difference in the average potential outcomes when all groups are treated versus when none are controlled. It is a widely studied estimand in settings with interference such as bipartite spatial experiments (Zigler and Papadogeorgou, 2021; Harshaw et al., 2023). Denote $\mu_1 = n^{-1} \sum_{i=1}^{n} Y_i(\mathbf{1})$ and $\mu_0 = n^{-1} \sum_{i=1}^{n} Y_i(\mathbf{0})$, we have $\tau = \mu_1 - \mu_0$.

3 Proposed estimation procedure

3.1 Point estimator

Let $T_i = \mathbb{1}\{\sum_{k=1}^{m} W_{ik}(1 - Z_k) = 0\}$ and $C_i = \mathbb{1}\{\sum_{k=1}^{m} W_{ik} Z_k = 0\}$ denote the indicator that all groups which unit i belongs to were assigned to the treatment group and the control group, respectively. Across the paper, we focus on Bernoulli randomization where each group is independently assigned to the treatment group with probability p and the control group with probability $1-p$, i.e., $Z_k \overset{\text{iid}}{\sim} \text{Bern}(p)$. Under Bernoulli randomization, we have $E(T_i) = p^{|\mathcal{S}_i|}$ and $E(C_i) = (1-p)^{|\mathcal{S}_i|}$. A natural Horvitz-Thompson-type estimator $n^{-1}\sum_{i=1}^n T_i Y_i / p^{|\mathcal{S}_i|} - n^{-1}\sum_{i=1}^n C_i Y_i/(1-p)^{|\mathcal{S}_i|}$ is unbiased to τ. However, throughout this paper, we focus on the following Hájek-type weighting estimator $\hat{\tau} = \hat{\mu}_1 - \hat{\mu}_0$, where

$$\begin{aligned}
\hat{\mu}_1 &= n^{-1}\sum_{i=1}^n \frac{T_i Y_i}{p^{|\mathcal{S}_i|}} \Big/ n^{-1}\sum_{i=1}^n \frac{T_i}{p^{|\mathcal{S}_i|}}, \\
\hat{\mu}_0 &= n^{-1}\sum_{i=1}^n \frac{C_i Y_i}{(1-p)^{|\mathcal{S}_i|}} \Big/ n^{-1}\sum_{i=1}^n \frac{C_i}{(1-p)^{|\mathcal{S}_i|}},
\end{aligned}$$

because of its better finite-sample performance compared with the unbiased Horvitz-Thompson estimator.

3.2 Consistency of $\hat{\tau}$

Assumption 2. *$\bar{S} = O(1)$ and $\bar{D}/n = o(1)$.*

Assumption 2 restricts the density of the bipartite graph as n increases. We restrict the maximum number of groups each unit belongs to bounded by a constant while allowing for the maximum number of units each group contains to increase with n but at a slower rate.

Assumption 3. *The potential outcomes and the covariates are bounded.*

We have the following consistency result for $\hat{\tau}$.

Theorem 1 (Consistency of $\hat{\tau}$)**.** *Under Assumptions 1–3, $\hat{\tau}$ converges in probability to τ.*

We introduce additional notation. Let $\boldsymbol{Y}(\boldsymbol{z}) = (Y_1(\boldsymbol{z}), \dots, Y_n(\boldsymbol{z}))$ denote the vector consisting all potential outcomes under treatment assignment $\boldsymbol{z}$, and $\tilde{\boldsymbol{Y}}(\boldsymbol{z}) = \boldsymbol{Y}(\boldsymbol{z}) - n^{-1}\sum_{i=1}^n Y_i(\boldsymbol{z})$ denote the centered potential outcome vector.

3.3 Asymptotic distribution

We further assume the following condition on the density of the bipartite graph to establish the asymptotic distribution of $\hat{\tau}$.

Assumption 4 (Sparse bipartite graph)**.** *Define groups j_1 and j_2 are connected if there exists at least one unit belonging to both groups. Assume for any group k, the total number of groups that are connected to k*

is bounded by an absolute constant B.

$$\sum_{j\in[m]\backslash\{k\}} \mathbb{1}\{j,k \text{ are connected}\} \le B.$$

Assumption 4 restricts how dense the bipartite graph could be.

Theorem 2 (Asymptotic distribution of $\hat{\tau}$)**.** *Under Assumptions 1–4, the variance of $\hat{\tau}$ has the order:* $\mathrm{var}(\hat{\tau})/v_n = 1 + o(1)$*, where*

$$v_n \quad = \quad n^{-2}\left\{\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}) + \tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}) + 2\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0})\right\}, \tag{1}$$

where for $i,j = 1,\ldots,n$,

$$(\Lambda_1)_{i,j} = p^{-|\mathcal{S}_i\cap\mathcal{S}_j|} - 1, \quad (\Lambda_0)_{i,j} = (1-p)^{-|\mathcal{S}_i\cap\mathcal{S}_j|} - 1, \quad (\Lambda_\tau)_{i,j} = \mathbb{1}\{\mathcal{S}_i\cap\mathcal{S}_j \ne \varnothing\}.$$

Further assuming the non-degeneracy of v_n, i.e., $v_n \asymp \bar{D}/n$, we have

$$v_n^{-1/2}(\hat{\tau}-\tau) \quad \to \quad \mathcal{N}(0,1)$$

in distribution.

Remark 1 (Non-degeneracy of v_n)**.** *In Theorem 2, we impose the condition that $v_n \asymp \bar{D}/n$. This is motivated by several aspects. First, in the supplementary material, we prove that under Assumptions 1–4, the variance has an upper bound*

$$v_n \le \frac{c\bar{D}}{n}, \tag{2}$$

where $c > 0$ is a universal constant. The non-degeneracy condition assumes the upper bound is tight. Inherently, it requires the potential outcomes to have a non-degenerate covariance structure. Second, in the special case of completely randomized experiments, $\bar{D} = 1$. Hence, the non-degeneracy condition serves as a generalization of the classical setting. More generally, our theory allows v_n to have a looser lower bound: $v_n \ge c'(\bar{D}/n)^p$ where $1 \le p < 3/2$.

As an additional sanity check, we justify the nondegeneracy condition in some random data examples. Consider a network where $\bar{D}$ is finite. If the potential outcomes $(Y_i(\mathbf{1}), Y_i(\mathbf{0}))$ are generated independently *from a bivariate normal distribution* $\mathcal{N}(\mathbf{0}_2, \sigma_i^2 \mathbf{I}_2)$*, then the quantities in* v_n *has the following order:*

$$n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}) = n^{-2}\sum_{i=1}^{n}(p^{-|\mathcal{S}_i|}-1)\sigma_i^2 + o(n^{-1}),$$
$$n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}) = n^{-2}\sum_{i=1}^{n}(p^{-|\mathcal{S}_i|}-1)\sigma_i^2 + o(n^{-1}),$$
$$n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0}) = o(n^{-1}),$$

which ensures that

$$v_n \asymp 2n^{-2}\sum_{i=1}^{n}(p^{-|\mathcal{S}_i|}-1)\sigma_i^2.$$

Below we use classical Bernoulli randomization and cluster randomization as examples to illustrate the variance formula in Theorem 2. Our Theorem 2 recovers the canonical results in the literature in these specific settings.

Example 5 (Classical Bernoulli randomized experiment)**.** *In classical Bernoulli randomized experiments,*

$$\mathcal{S}_i \cap \mathcal{S}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Thus the asymptotic variance in equation (1) *reduces to*

$$v_n = n^{-2}p(1-p)\sum_{i=1}^{n}\left\{\frac{\tilde{Y}_i(1)}{p} - \frac{\tilde{Y}_i(0)}{1-p}\right\}^2.$$

This variance formula recovers the classical result by Neyman (1990).

Example 6 (Cluster randomization)**.** *In a cluster randomization setting with* m *clusters and the treatment assignment* $Z_k \overset{iid}{\sim} \mathrm{Bern}(p)$ *for* $k = 1, \ldots, m$*, we have*

$$\mathcal{S}_i \cap \mathcal{S}_j = \begin{cases} 1, & \text{if } i, j \text{ belong to the same group}, \\ 0, & \text{otherwise}. \end{cases}$$

If we order the units according to the cluster they belong to,

$$\Lambda_\tau = \begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{n_m} \end{pmatrix}, \quad \Lambda_1 = \frac{1-p}{p}\Lambda_\tau, \quad \Lambda_0 = \frac{p}{1-p}\Lambda_\tau,$$

where $\mathbf{1}_{n_k}$ is an $n_k \times n_k$-dimensional matrix with all entries equal to 1 and n_k is the total number of units in cluster k for $k = 1, \ldots, m$. Therefore, the asymptotic variance in equation (1)reduces to

$$v_n = n^{-2}p(1-p)\sum_{k=1}^{m}\left[\sum_{i=1}^{n_k}\left\{\frac{\tilde{Y}_i(1)}{p} - \frac{\tilde{Y}_i(0)}{1-p}\right\}\right]^2.$$

This variance formula recovers the result in Su and Ding (2021).

3.4 Variance estimation

We propose the following variance estimator for $\mathrm{avar}(\hat{\tau})$:

$$\hat{v} = \left[\left\{n^{-2}\sum_{i,j}\frac{T_iT_j(Y_i - \hat{\mu}_1)(Y_j - \hat{\mu}_1)(\Lambda_1)_{i,j}}{p^{|S_i \cup S_j|}}\right\}^{1/2} + \left\{n^{-2}\sum_{i,j}\frac{C_iC_j(Y_i - \hat{\mu}_0)(Y_j - \hat{\mu}_0)(\Lambda_0)_{i,j}}{(1-p)^{|S_i \cup S_j|}}\right\}^{1/2}\right]^2 \quad (3)$$

The following Theorem 3 shows that $\hat{v}$ converges in probability and is conservative to the true asymptotic variance $\mathrm{avar}(\hat{\tau})$. Therefore, we can construct a confidence interval: $[\hat{\tau} - q_{\alpha/2}\hat{v}^{1/2}, \hat{\tau} + q_{\alpha/2}\hat{v}^{1/2}]$ that ensures valid Type I error control in a large sample, where $q_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of a standard Gaussian distribution.

Theorem 3 (Conservative variance estimator for $\hat{\tau}$)**.** *Under Assumptions 1–4,*

(a) $\hat{v}/\mathrm{plim}(\hat{v})$ *converges in probability to 1, where*

$$\mathrm{plim}(\hat{v}) = \left[\left\{n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1})\right\}^{1/2} + \left\{n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0})\right\}^{1/2}\right]^2.$$

(b) $\mathrm{plim}(\hat{v}) \geq \mathrm{avar}(\hat{\tau})$*, where the equality holds if and only if*

$$\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0}) = \{\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1})\}^{1/2}\{\tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0})\}^{1/2}. \quad (4)$$

The condition for the conservative variance estimator to be consistent, as shown in Theorem 3(b) is generally complicated to analyze. We revisit the two Examples 5 and 6 to provide more intuition of the equality condition in special cases.

Continuance of Example 5 (Classical Bernoulli randomized experiment). *In the classical Bernoulli randomized experiment setting, condition* (4) *reduces to*

$$\sum_{i=1}^{n} \tilde{Y}_i(1)\tilde{Y}_i(0) = \left\{ \sum_{i=1}^{n} \tilde{Y}_i(1)^2 \sum_{i=1}^{n} \tilde{Y}_i(0)^2 \right\}^{1/2},$$

which is equivalent to $\tilde{Y}_i(1) = \zeta_1 \tilde{Y}_i(0)$ *for any* $i = 1, \ldots, n$ *and* $\zeta_1 > 0$ *is a positive constant. One special case that satisfies this condition is the constant treatment effect case.*

Continuance of Example 6 (Cluster randomization). *In the cluster experiment setting, condition* (4) *reduces to*

$$\sum_{k=1}^{m} \left\{ \sum_{i=1}^{n_k} \tilde{Y}_i(1) \right\} \left\{ \sum_{i=1}^{n_k} \tilde{Y}_i(0) \right\} = \left[\sum_{k=1}^{m} \left\{ \sum_{i=1}^{n_k} \tilde{Y}_i(1) \right\}^2 \sum_{k=1}^{m} \left\{ \sum_{i=1}^{n_k} \tilde{Y}_i(0) \right\}^2 \right]^{1/2},$$

which is equivalent to $\sum_{i=1}^{n_k} \tilde{Y}_i(1) = \zeta_2 \sum_{i=1}^{n_k} \tilde{Y}_i(0)$ *for any* $k = 1, \ldots, m$ *and* $\zeta_2 > 0$ *is a positive constant.*

4 Covariate adjustment

4.1 Methodology

Let $\tilde{X}_i$ denote the centered covariates, i.e., $\tilde{X}_i = X_i - n^{-1}\sum_{i=1}^{n} X_i$. Consider a class of covariate-adjusted estimators indexed by (β_1, β_0)

$$\hat{\tau}(\beta_1, \beta_0) \quad = \quad n^{-1}\sum_{i=1}^{n} \frac{T_i(Y_i - \beta_1^{\mathrm{T}}\tilde{X}_i)}{p^{|\mathcal{S}_i|}} \Big/ n^{-1}\sum_{i=1}^{n} \frac{T_i}{p^{|\mathcal{S}_i|}} - n^{-1}\sum_{i=1}^{n} \frac{C_i(Y_i - \beta_0^{\mathrm{T}}\tilde{X}_i)}{(1-p)^{|\mathcal{S}_i|}} \Big/ n^{-1}\sum_{i=1}^{n} \frac{C_i}{(1-p)^{|\mathcal{S}_i|}},$$

where we replace Y_i in the naive estimator $\hat{\tau}$ with linearly adjusted residuals. Further denote $\tilde{\boldsymbol{X}} = (\tilde{X}_1, \ldots, \tilde{X}_n)^{\mathrm{T}}$ the centered covariate matrix including all n units. The covariate adjustment estimator $\hat{\tau}(\beta_1, \beta_0)$ has the following properties for fixed (β_1, β_0).

Proposition 1 (Consistency and asymptotic distribution of $\hat{\tau}(\beta_1, \beta_0)$). *Under Assumptions 1–3, for any fixed* (β_1, β_0)*,* $\hat{\tau}(\beta_1, \beta_0)$ *converges in probability to* τ*. Further suppose Assumption 4 holds, the variance of*

$\hat{\tau}(\beta_1, \beta_0)$ has the order $\text{var}\{\hat{\tau}(\beta_1, \beta_0)\}/v_n(\beta_1, \beta_0) = 1 + o(1)$, *where*

$$\begin{aligned} v_n(\beta_1, \beta_0) &= n^{-2}\Big[\{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\beta_1\}^{\mathrm{T}}\Lambda_1\{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\beta_1\} + \{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\beta_0\}^{\mathrm{T}}\Lambda_0\{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\beta_0\} \\ &\quad + 2\{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\beta_1\}^{\mathrm{T}}\Lambda_\tau\{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\beta_0\}\Big]. \end{aligned} \tag{5}$$

Further assuming the non-degeneracy of $v_n(\beta_1, \beta_0)$, i.e., $v_n(\beta_1, \beta_0) \asymp \bar{D}/n$, we have

$$v_n(\beta_1, \beta_0)^{-1/2}\{\hat{\tau}(\beta_1, \beta_0) - \tau(\beta_1, \beta_0)\} \to \mathcal{N}(0, 1)$$

in distribution.

Proposition 1 states the analogous results to Theorem 2 on the asymptotic distribution of the class of covariate-adjusted estimators. The results follow directly when we treat the linearly adjusted residuals of the potential outcomes, $Y_i(1) - \beta_1^{\mathrm{T}}\tilde{X}_i$ and $Y_i(0) - \beta_0^{\mathrm{T}}\tilde{X}_i$, as "pseudo potential outcomes" and apply Theorem 2. Similarly, we can construct a conservative variance estimator for $\hat{\tau}(\beta_1, \beta_0)$. Denote the upper bound of $v_n(\beta_1, \beta_0)$ as

$$v_{n,\mathrm{UB}}(\beta_1, \beta_0) = \left(\Big[n^{-2}\{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\beta_1\}^{\mathrm{T}}\Lambda_1\{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\beta_1\}\Big]^{1/2} + \Big[n^{-2}\{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\beta_0\}^{\mathrm{T}}\Lambda_0\{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\beta_0\}\Big]^{1/2}\right)^2$$

and the corresponding consistent estimator of the upper bound as

$$\begin{aligned} \hat{v}_{n,\mathrm{UB}}(\beta_1, \beta_0) &= \left[\left\{\sum_{i,j} \frac{T_iT_j(Y_i - \hat{\mu}_1 - \beta_1^{\mathrm{T}}\tilde{X}_i)(Y_j - \hat{\mu}_1 - \beta_1^{\mathrm{T}}\tilde{X}_j)(\Lambda_1)_{i,j}}{p^{|S_i \cup S_j|}}\right\}^{1/2}\right. \\ &\quad \left. + \left\{\sum_{i,j} \frac{C_iC_j(Y_i - \hat{\mu}_0 - \beta_0^{\mathrm{T}}\tilde{X}_i)(Y_j - \hat{\mu}_0 - \beta_0^{\mathrm{T}}\tilde{X}_j)(\Lambda_0)_{i,j}}{(1-p)^{|S_i \cup S_j|}}\right\}^{1/2}\right]^2. \end{aligned}$$

To gain the best asymptotic efficiency by using covariate adjustment estimators, ideally, we want to minimize the asymptotic variance in (5) across values of (β_1, β_0). However, the third term in (5) is not estimable because it depends on the joint distribution of the potential outcomes. Instead, the improvement in the asymptotic variance of the covariate-adjusted estimator, $\hat{\tau}(\beta_1, \beta_0)$, compared with that of the naive estimator $\hat{\tau}$, is estimable. Let $L(\beta_1, \beta_0)$ denote the difference in the two asymptotic variances, i.e.,

$$L(\beta_1, \beta_0) = v_n(\beta_1, \beta_0) - v_n(0, 0)$$

$$= \begin{pmatrix}\beta_1\\ \beta_0\end{pmatrix}^{\mathrm{T}} \begin{pmatrix}\tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}}\\ \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{X}}\end{pmatrix} \begin{pmatrix}\beta_1\\ \beta_0\end{pmatrix} - 2\begin{pmatrix}\tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}) + \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0})\\ \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}) + \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{1})\end{pmatrix}^{\mathrm{T}} \begin{pmatrix}\beta_1\\ \beta_0\end{pmatrix}.$$

Define the optimization problem and its solution as

$$\left(\tilde{\beta}_1, \tilde{\beta}_0\right) = \arg\min_{\beta_1,\beta_0} L(\beta_1, \beta_0). \tag{6}$$

By construction, the improvement in asymptotic variance is guaranteed. We formalize this result in the following proposition.

Proposition 2 (Improvement in asymptotic variance)**.** *The covariate adjustment estimator $\hat{\tau}(\tilde{\beta}_1, \tilde{\beta}_0)$ has an asymptotic variance no larger that of $\hat{\tau}$, i.e., $v_n(\tilde{\beta}_1, \tilde{\beta}_0) \le v_n$.*

4.2 Estimation and inference based on covariate adjustment

The optimization in (6) is a population-level convex problem which has a closed-form global optimal solution $(\tilde{\beta}_1, \tilde{\beta}_0)$,

$$\begin{pmatrix}\tilde{\beta}_1\\ \tilde{\beta}_0\end{pmatrix} = \begin{pmatrix}\tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}}\\ \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{X}}\end{pmatrix}^{-1} \begin{pmatrix}\tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}) + \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0})\\ \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}) + \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{1})\end{pmatrix}.$$

We propose to estimate the vector which includes unobserved potential outcomes using the sample analog by inverse probability weighting, similar to the trick used for variance estimation in Section 3.4. We construct the following estimator of $(\tilde{\beta}_1, \tilde{\beta}_0)$,

$$\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_0\end{pmatrix} = \begin{pmatrix}\tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}}\\ \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{X}}\end{pmatrix}^{-1} \begin{pmatrix}\sum_{i,j}\frac{T_iT_j\tilde{X}_i(Y_j-\hat{\mu}_1)(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}} + \sum_{i,j}\frac{C_iC_j\tilde{X}_i(Y_j-\hat{\mu}_0)(\Lambda_\tau)_{i,j}}{(1-p)^{|S_i\cup S_j|}}\\ \sum_{i,j}\frac{T_iT_j\tilde{X}_i(Y_j-\hat{\mu}_1)(\Lambda_\tau)_{i,j}}{p^{|S_i\cup S_j|}} + \sum_{i,j}\frac{C_iC_j\tilde{X}_i(Y_j-\hat{\mu}_0)(\Lambda_0)_{i,j}}{(1-p)^{|S_i\cup S_j|}}\end{pmatrix}.$$

We next establish the asymptotic properties of the covariate-adjusted estimator $\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0)$. To simplify the discussion, we introduce the following assumption that imposes the limits for several population quantities.

Assumption 5 (Existence of limiting values)**.** *Assume that*

$$(\bar{D}/n)\left(\tilde{\boldsymbol{Y}}(1) \quad \tilde{\boldsymbol{X}}\right)^{\mathrm{T}}\Lambda_1\left(\tilde{\boldsymbol{Y}}(1) \quad \tilde{\boldsymbol{X}}\right) \to \begin{pmatrix}\Omega_{yy,11} & \Omega_{yx,11}\\ \Omega_{yx,11}^{\mathrm{T}} & \Omega_{xx,11}\end{pmatrix} =: \Omega_{11},$$

$$(\bar{D}/n)\begin{pmatrix}\tilde{\boldsymbol{Y}}(0) & \tilde{\boldsymbol{X}}\end{pmatrix}^{\mathrm{T}}\Lambda_0\begin{pmatrix}\tilde{\boldsymbol{Y}}(0) & \tilde{\boldsymbol{X}}\end{pmatrix} \rightarrow \begin{pmatrix}\Omega_{yy,00} & \Omega_{yx,00}\\ \Omega_{yx,00}^{\mathrm{T}} & \Omega_{xx,00}\end{pmatrix} =: \Omega_{00},$$

$$(\bar{D}/n)\begin{pmatrix}\tilde{\boldsymbol{Y}}(1) & \tilde{\boldsymbol{X}}\end{pmatrix}^{\mathrm{T}}\Lambda_\tau\begin{pmatrix}\tilde{\boldsymbol{Y}}(0) & \tilde{\boldsymbol{X}}\end{pmatrix} \rightarrow \begin{pmatrix}\Omega_{yy,10} & \Omega_{yx,10}\\ \Omega_{yx,01}^{\mathrm{T}} & \Omega_{xx,10}\end{pmatrix} =: \Omega_{10}.$$

Assumption 5 requires the weighted covariance matrices of potential outcomes and covariates to have limiting values not depending on n as $n \to \infty$. In the special case of complete randomized experiments without interference, it reduces to the assumption in Theorem 5 in Li and Ding (2017).

The following theorem shows the consistency and asymptotic distribution of $\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0)$.

Theorem 4 (Consistency and asymptotic distribution of $\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0)$)**.** *Under Assumptions 1–3 and 5, $\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0)$ converges in probability to τ. Further suppose Assumption 4 holds,*

$$\left[\mathrm{avar}\{\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0\}\right]^{-1/2}\left\{\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0) - \tau\right\} \rightarrow \mathcal{N}(0,1)$$

in distribution.

Combined with Proposition 2, Theorem 4 suggests the covariate-adjusted estimator can reduce the asymptotic variance compared with the unadjusted estimator.

Now following our discussion in Section 4.1, we can use the variance estimator $\hat{v}_{n,\mathrm{UB}}(\hat{\beta}_1, \hat{\beta}_0)$ by plugging in the estimated coefficients. The following theorem establishes the convergence and conservativeness of this variance estimator.

Theorem 5 (Conservative variance estimator for $\hat{\tau}^{\mathrm{adj}}$)**.** *The variance estimator $\hat{v}_{n,\mathrm{UB}}(\hat{\beta}_1, \hat{\beta}_0)$ is a conservative variance estimator following the facts that $\hat{v}_{n,\mathrm{UB}}(\hat{\beta}_1, \hat{\beta}_0)/v_{n,\mathrm{UB}}(\tilde{\beta}_1, \tilde{\beta}_0)$ converges in probability to 1 and that $v_{n,\mathrm{UB}}(\tilde{\beta}_1, \tilde{\beta}_0) \geq \mathrm{avar}\{\hat{\tau}(\tilde{\beta}_1, \tilde{\beta}_0)\}$.*

Theorem 5 proves the conservativeness of the variance estimator, which directly motivates a valid confidence interval: $[\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0) - q_{\alpha/2}\hat{v}_{n,\mathrm{UB}}^{1/2}, \hat{\tau}(\hat{\beta}_1, \hat{\beta}_0) + q_{\alpha/2}\hat{v}_{n,\mathrm{UB}}^{1/2}]$.

5 Simulation and real-world data analysis

5.1 Simulated bipartite graph

We conduct simulation studies to evaluate the finite sample performance of our proposed estimators in this section. In the simulation, we start by creating two types of nodes: individual nodes and group nodes.

We generate the degrees of individual nodes following Gaussian distribution and truncate them to ensure they are between 0 and $\bar{S}$. For each node i, we randomly assign a degree $|\mathcal{S}_i|$ within this range. Next, we randomly connect individual nodes i to $|\mathcal{S}_i|$ different groups from the set of group nodes. Moreover, we make the following adjustments to ensure the graph satisfies the sparsity condition in Assumption 4. For each degree s, we examine the number of connected group sets through individuals with degree s. If the count surpasses a predefined upper limit, we break a random subset of the connections. Specifically, we break links between individuals and groups that are part of the same overly connected set. We then randomly establish new connections with other group sets that were not previously overly connected, ensuring that the total degree of each individual is unchanged. We keep the bipartite graph fixed after building it up.

We consider three different regimes of data generating process. For each regime, we generate covariates $X_i = (X_{1i}, X_{2i}) \sim (U[0,1])^2$ and the potential outcomes $Y_i(\mathbf{1})$ and $Y_i(\mathbf{0})$ from the following conditional distributions summarized in Table 1, with $\gamma = (1,1)^{\mathrm{T}}$ and $\alpha_i \sim U[0, 0.5]$. The treatment indicator $Z_k \stackrel{\text{iid}}{\sim} \text{Bern}(p)$ with $p = 0.5$.

Regime	$Y_i(\mathbf{1})$	$Y_i(\mathbf{0})$
R1	$\mathcal{N}(0.25 + \gamma^{\mathrm{T}} X_i, 1)$	$\mathcal{N}(\gamma^{\mathrm{T}} X_i, 1)$
R2	$\mathcal{N}(\alpha_i + \gamma^{\mathrm{T}} X_i, 1)$	$\mathcal{N}(\gamma^{\mathrm{T}} X_i, 1)$
R3	$\mathcal{N}(0.1\vert\mathcal{S}_i\vert + 1.1\gamma^{\mathrm{T}} X_i, 1.5)$	$\mathcal{N}(\gamma^{\mathrm{T}} X_i, 1.5)$

Table 1: Three regimes of data generating process

In Table 2, we report the finite sample performance of the two estimators $\hat{\tau}$ and $\hat{\tau}^{\text{adj}}$ with $n = 5000$, $m = 1500$, and $\bar{S} = 5$ based on 1000 Monte Carlo replications. In all three regimes, the two estimators both have small finite sample bias, and the proposed variance estimators are conservative, leading to valid inference with coverage rates larger than 95%. Compared with the naive estimator $\hat{\tau}$, the covariate-adjusted estimator has s smaller standard error and higher power under all regimes. Although our theory guarantees efficiency improvement only in asymptotic variance, in the numerical studies, we also observe smaller variance estimators and thus shorter constructed confidence intervals under all three regimes.

5.2 Data analysis based on real-world bipartite graph

In this section, we apply our estimators to analyze a real-world bipartite graph. We revisit the application discussed in Zigler and Papadogeorgou (2021) and Papadogeorgou et al. (2019), which studies the causal

		naive estimator					covariate adjustment				
Regime	τ	$\hat{\tau}$	$\text{SE}(\hat{\tau})$	$\hat{\text{SE}}(\hat{\tau})$	coverage	power	$\hat{\tau}^{\text{adj}}$	$\text{SE}(\hat{\tau}^{\text{adj}})$	$\hat{\text{SE}}(\hat{\tau}^{\text{adj}})$	coverage	power
R1	0.221	0.223	0.059	0.086	99.7%	82.3%	0.223	0.055	0.080	99.5%	89.3%
R2	0.256	0.255	0.062	0.085	98.8%	92.8%	0.254	0.058	0.079	98.8%	96.0%
R3	0.355	0.358	0.085	0.124	99.6%	90.6%	0.358	0.082	0.119	99.5%	93.4%

Table 2: Finite sample performance of estimators $\hat{\tau}$ and $\hat{\tau}^{\text{adj}}$. For each regime of data generating process, we report the true total treatment effect τ, the two point estimators, their standard error $\text{SE}(\cdot)$, standard error estimator $\hat{\text{SE}}(\cdot)$, the coverage rate of the 95% confidence interval constructed using the conservative variance estimator, and their power.

effect of the installation of selective non-catalytic reduction system at a power plant on the air pollution level in the nearby areas. The intervention is at the power plant level, i.e., each power plant may be assigned to the implementation of the new system ($Z_k = 1$) or not ($Z_k = 0$). Since multiple power plants simultaneously influence a given area, and each power plant can potentially affect multiple areas, it forms a natural bipartite graph. To model the scenario, we simulate a bipartite randomized experiment based on the real-world bipartite structure between power plants and nearby areas. We take power plants as treatment units and air pollution monitors as outcome units.

We construct our dataset using the power plant dataset from Papadogeorgou et al. (2019) and 2004 air pollution data at the monitor level from the United States Environmental Protection Agency's website. Additionally, we incorporate population information for the counties where the monitors are located. The initial dataset of outcome units includes 95,762 air quality monitors, and the intervention units correspond to 473 coal or natural gas-burning power plants operating in the continental U.S. during the summer of 2004. To prepare the dataset, we remove outcome units with an "Arithmetic Mean" above the 90th percentile among all observations, and we also exclude outcome units with an "Arithmetic Mean" around 0 (we choose 2 as the threshold in this application), and outcome units with a population size exceeding 10^6. To address computational constraints, we randomly select 10% of the remaining monitors. We calculate the distances between monitors and power plants using their longitude and latitude coordinates. A bipartite graph is then constructed by connecting monitors to power plants located within 15 km. Finally, we remove monitors and power plants that are not connected to any other units, resulting in a dataset comprising 795 outcome units and 228 intervention units. The maximum degree of outcome units is restricted to be 2.

We assume the potential outcomes to be $Y_i(\mathbf{1}) = \gamma_1^{\text{T}} X_i + \varepsilon_1$ and $Y_i(\mathbf{0}) = \gamma_0^{\text{T}} X_i + \varepsilon_0$, where $\gamma_1 = (2, -2, -2)^{\text{T}}$, $\gamma_0 = (1, -1, -1)^{\text{T}}$, and $\varepsilon_1, \varepsilon_0 \sim U[0, 15]$. This data generation process is designed to simulate the distribution of the observed 'Arithmetic Mean' in the pollution dataset. To standardize the covariates and ensure numerical stability, we scale the population seize of the county where the monitor i is located by dividing it by 10^6 (X_{1i}), and the distance between monitor i and its closest power plant by dividing it by 30 (X_{2i}). The third covariate, X_{3i}, represents the number of power plants connected to monitor i. We consider 1000 Monte Carlo replications. In each replication, treatment units are randomly assigned to treatment with a probability of $p = 0.5$. When applying the covariate adjustment estimator, we include the scaled covariates X_{1i}, X_{2i}, and X_{3i} in the model. The true total treatment effect is -1.266.

Table 3 below reports the simulation results based on the real-world bipartite graph. We can see that both the naive estimator and the covariate-adjusted estimator have small biases for estimating the true treatment effect, and both strategies lead to valid yet slightly conservative confidence intervals. Nevertheless, by applying the covariate adjustment strategy we introduced in Section 4, we can witness a reduction in both the standard deviation of the point estimator and the estimated variance, which leads to a great improvement in the power.

estimator	point estimator	SE	$\hat{\text{SE}}$	coverage	power
naive estimator $\hat{\tau}$	-1.251	0.136	0.227	98.2%	80.6%
covariate adjustment $\hat{\tau}^{\text{adj}}$	-1.202	0.116	0.170	97.7%	86.3%

Table 3: Simulation results based on real bipartite graph in the power plant application. We report two point estimators (with and without covariate adjustment), their standard error SE, standard error estimator $\hat{\text{SE}}$, the coverage rate of the 95% confidence interval constructed using the conservative variance estimator, and their power.

6 Discussion

In this work, we propose a design-based causal inference framework for bipartite experiments. We generalize the classical SUTVA to the bipartite experiment setting and provide point and variance estimators for estimating the total treatment effect. These estimators are based on theoretical results that guarantee the consistency and asymptotic normality of the point estimator and the conservativeness of the variance estimator. We also propose covariate adjustment strategies that help improve the efficiency of the point estimator. This framework serves as a general extension of design-based causal inference frameworks for completely randomized experiments and cluster randomized experiments.

While this framework is useful for estimating causal effects in many general scenarios involving bipartite experiments, there are several directions for further investigation. First, we focus on the total average treatment effect which compares all versus nothing treatment regimes. There are more general causal parameters of interest that we can explore. Second, we only discuss the Bernoulli randomization treatment regime, leaving other more complex bipartite intervention strategies undeveloped. Third, we mainly focus on the outcome unit-level covariates for the covariate adjustment strategy. When treatment unit-level covariates are also available, as an ad-hoc strategy, we can incorporate them by using a summary at the outcome-unit level, for instance, taking the average or sum of the covariate values for the groups that each unit is connected to. However, a more rigorous and systematic way of incorporating treatment-unit level covariates is unclear. We leave them for future research.

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Supplementary Material

Section S1 provides a general theory of establishing central limit theorems for bipartite experiments under Bernoulli randomization.

Section S2 provides proofs of all theorems in the main text.

Section S3 provides additional results on the special case when $\bar{S} = 2$.

S1 Theory under Bernoulli Randomization

We first study the distribution of a general statistic defined as follows:

$$\Gamma = \sum_{k_1} a_{k_1} \tilde{Z}_{k_1} + \sum_{k_1 < k_2} a_{k_1 k_2} \tilde{Z}_{k_1} \tilde{Z}_{k_2} + \cdots + \sum_{k_1 < \cdots < k_{\bar{S}}} a_{k_1 \ldots k_{\bar{S}}} \tilde{Z}_{k_1} \cdots \tilde{Z}_{k_{\bar{S}}}. \tag{S1}$$

Here $\{a_{k_1 \ldots k_s} : k_1, \ldots, k_s \in [m],\ k_1 \neq \ldots \neq k_s\}$ is an s-dimensional array that are symmetric in its indices, i.e.,

$$a_{k_1 \ldots k_s} = a_{k_1' \ldots k_s'} \quad \text{if} \quad \{k_1, \ldots, k_s\} = \{k_1', \ldots, k_s'\}.$$

As a convention, we use $(k_1 \ldots k_s)$ to denote an unordered s-tuple with $k_1 \neq \cdots \neq k_s$. Moreover, $\tilde{Z}_k$'s are i.i.d. copies of a random variable $\tilde{Z}$ with mean zero, variance σ^2 and fourth moments bounded by $E\tilde{Z}^4 \leq \nu_4^4$. Note that here we do not require $\tilde{Z}$ to be a centered Bernoulli variable.

For the statistic in (S1), we have $E(\Gamma) = 0$ and

$$v_\Gamma = \mathrm{var}(\Gamma) = \sum_{k_1} a_{k_1}^2 \sigma^2 + \sum_{k_1 < k_2} a_{k_1 k_2}^2 \sigma^4 + \cdots + \sum_{k_1 < \cdots < k_{\bar{S}}} a_{k_1 \ldots k_{\bar{S}}}^2 \sigma^{2\bar{S}}.$$

We have the following central limit theorem for Γ:

Theorem S1. *Assume that*

1. *the array a's are bounded by some constant $\bar{a}_m$ that possibly depends on m;*
2. *there exists a universal constant B such that for all $k \in [m]$ and $s \in [\bar{S}]$,*

$$\sum_{(k_1, \ldots, k_s) \subset [m] \setminus \{k\}} \mathbb{1}\{|a_{k k_1 \ldots k_s}| \neq 0\} \leq B;$$

3. *the variance is nondegenerate: $v_\Gamma / (m^{1/2} \bar{a}_m^2)$ goes to ∞.*

We have $v_\Gamma^{-1/2}\Gamma \to \mathcal{N}(0,1)$ in distribution.

We will use the martingale central limit theorem in Hall and Heyde (2014) to prove Theorem S1. For completeness of our proof, we first present the martingale central limit theorem as the following Proposition S1.

Proposition S1 (Theorem 3.2 of Hall and Heyde (2014))**.** *Let $\{S_{ni}, \mathscr{F}_{ni}, 1 \le i \le k_n, n \ge 1\}$ be a zero-mean, square-integrable martingale array with differences Δ_{ni}, and let η^2 be an almost surely finite random variable. Suppose the following conditions hold:*

1. *Squared sum convergence:*

$$\sum_i E(\Delta_{ni}^2 \mid \mathscr{F}_{n,i-1}) \quad \to \quad \eta^2 \tag{S2}$$

   in probability,

2. *Lindeberg condition:*

$$\text{for all } \varepsilon > 0, \quad \sum_i E(\Delta_{ni}^2 \mathbb{1}\{|\Delta_{ni}| > \varepsilon\} \mid \mathscr{F}_{n,i-1}) \quad \to \quad 0 \tag{S3}$$

   in probability,

and the σ-fields are nested: $\mathscr{F}_{n,i} \subseteq \mathscr{F}_{n+1,i}$, for $1 \le i \le k_n$, $n \ge 1$. Then $S_{nk_n} = \sum_i \Delta_{ni}$ converges in distribution (stably) to some random variable with characteristics function $E\{\exp\left(-\frac{1}{2}\eta^2 t^2\right)\}$.

In particular, a sufficient condition for the Lindeberg condition (S3) is given by the following Lyapunov condition:

$$\text{For some } \delta > 0, \quad \sum_{i=1}^{n} E\{|\Delta_{ni}|^{2+\delta}\} \to 0. \tag{S4}$$

Proof of Theorem S1. We prove Theorem S1 following three steps. We first construct a martingale difference sequence based on Γ. Next, we check the convergence of the summation of the conditional squared differences in equation (S2). Finally, we check the Lyapunov condition in equation (S4).

Step 1. Construct a martingale difference sequence based on Γ. Let $\mathscr{F}_{m,k}$ be the σ-algebra generated by $\tilde{Z}_1, \dots, \tilde{Z}_k$, i.e. $\mathscr{F}_{m,k} = \sigma\{\tilde{Z}_1, \dots, \tilde{Z}_k\}$. For ease of notation, for any $k_1, \dots, k_\ell \in [m]$, we denote $\tilde{Z}_{k_1 \dots k_\ell} = \tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}$. Let

$$\Delta_{mk} \quad = \quad v_\Gamma^{-1/2} \sum_{s=1}^{\bar{S} \wedge k} \sum_{(k_1 \dots k_{s-1}) \subset [k-1]} a_{k_1 \dots k_{s-1} k} \tilde{Z}_{k_1 \dots k_{s-1} k},$$

with $a_{\varnothing k} = a_k$ and $\tilde{Z}_{\varnothing} = 1$. Then $\{\Delta_{mk}, \mathscr{F}_{m,k}\}_{k=1}^m$ forms a martingale difference sequence and

$$\Gamma \quad = \quad \sum_{k=1}^{m} \Delta_{mk}.$$

Step 2. Check the convergence of the summation of the conditional squared differences in equation (S2)**.** We show equation S2 by computing the variance of its LHS,

$$
\begin{aligned}
& \operatorname{var}\left\{\sum_k E(\Delta_{mk}^2 \mid \mathscr{F}_{m,k-1})\right\} \\
= \; & \frac{\sigma^4}{v_\Gamma^2} \operatorname{var}\left(\sum_k \sum_{s,r}^{\bar{S}\wedge k} \sum_{\substack{(k_1\ldots k_s)\subset[k-1] \\ (k_1'\ldots k_r')\subset[k-1]}} a_{k_1\ldots k_s k} a_{k_1'\ldots k_r' k} \tilde{Z}_{k_1\ldots k_s} \tilde{Z}_{k_1'\ldots k_r'} \right) \\
= \; & \frac{\sigma^4}{v_\Gamma^2} \left\{ \sum_{k,\ell} \sum_{s,r}^{\bar{S}\wedge k} \sum_{t,u}^{\bar{S}\wedge \ell} \sum_{\substack{(k_1\ldots k_s)\subset[k-1] \\ (k_1'\ldots k_r')\subset[k-1] \\ (\ell_1'\ldots \ell_t')\subset[\ell-1] \\ (\ell_1'\ldots \ell_u')\subset[\ell-1]}} a_{k_1\ldots k_s k} a_{k_1'\ldots k_r' k} a_{\ell_1\ldots \ell_t \ell} a_{\ell_1'\ldots \ell_u' \ell} \operatorname{cov}(\tilde{Z}_{k_1\ldots k_s}\tilde{Z}_{k_1'\ldots k_r'}, \tilde{Z}_{\ell_1\ldots \ell_t}\tilde{Z}_{\ell_1'\ldots \ell_u'}) \right\} \qquad \text{(S5)}
\end{aligned}
$$

Note that $\operatorname{cov}(\tilde{Z}_{k_1\ldots k_s}\tilde{Z}_{k_1'\ldots k_r'}, \tilde{Z}_{\ell_1\ldots \ell_t}\tilde{Z}_{\ell_1'\ldots \ell_u'}) \neq 0$ only if

$$\{(k_1\ldots k_s)\cup(k_1'\ldots k_r')\} \cap \{(\ell_1\ldots \ell_t)\cup(\ell_1'\ldots \ell_u')\} \quad \neq \quad \varnothing.$$

For the nonzero covariance, we have

$$
\begin{aligned}
& \left| \operatorname{cov}(\tilde{Z}_{k_1\ldots k_s}\tilde{Z}_{k_1'\ldots k_r'}, \tilde{Z}_{\ell_1\ldots \ell_t}\tilde{Z}_{\ell_1'\ldots \ell_u'}) \right| \\
\leq \; & \left\{ \operatorname{var}(\tilde{Z}_{k_1\ldots k_s}\tilde{Z}_{k_1'\ldots k_r'}) \operatorname{var}(\tilde{Z}_{\ell_1\ldots \ell_t}\tilde{Z}_{\ell_1'\ldots \ell_u'}) \right\}^{1/2} \\
\leq \; & \left\{ E(Z_{k_1\ldots k_s}^2 Z_{k_1'\ldots k_r'}^2) E(Z_{\ell_1\ldots \ell_t}^2 Z_{\ell_1'\ldots \ell_u'}^2) \right\}^{1/2} \\
\leq \; & E(\tilde{Z}_{k_1\ldots k_s}^4)^{1/4} E(\tilde{Z}_{k_1'\ldots k_r'}^4)^{1/4} E(\tilde{Z}_{\ell_1\ldots \ell_t}^4)^{1/4} E(\tilde{Z}_{\ell_1'\ldots \ell_u'}^4)^{1/4} \\
\leq \; & \nu_4^{s+r+t+u} \leq \nu_4^{4(\bar{S}-1)}.
\end{aligned}
$$

Therefore, we can further bound (S5) as

$$\begin{aligned}
&\operatorname{var}\left\{\sum_k E(\Delta_{mk}^2 \mid \mathscr{F}_{m,k-1})\right\} \\
\leq\ & \frac{\sigma^4 \nu_4^{4(\bar{S}-1)} \bar{a}_m^4}{v_\Gamma^2} \left(\sum_{\substack{G_1,G_2,G_3,G_4 \subset [\bar{S}]: \\ G_1 \cap G_2 \neq \varnothing, \\ G_2 \cap G_3 \neq \varnothing, \\ G_3 \cap G_4 \neq \varnothing}} \mathbb{1}\{|a_{G_1}| \neq 0\} \mathbb{1}\{|a_{G_2}| \neq 0\} \mathbb{1}\{|a_{G_3}| \neq 0\} \mathbb{1}\{|a_{G_4}| \neq 0\} \right) \\
\leq\ & \frac{\sigma^4 \nu_4^{4(\bar{S}-1)} \bar{a}_m^4 (B\bar{S}^2)}{v_\Gamma^2} \left(\sum_{\substack{G_1,G_2,G_3 \subset [\bar{S}]: \\ G_1 \cap G_2 \neq \varnothing, \\ G_2 \cap G_3 \neq \varnothing}} \mathbb{1}\{|a_{G_1}| \neq 0\} \mathbb{1}\{|a_{G_2}| \neq 0\} \mathbb{1}\{|a_{G_3}| \neq 0\} \right) \\
\leq\ & \frac{\sigma^4 \nu_4^{4(\bar{S}-1)} \bar{a}_m^4 (B^2\bar{S}^4)}{v_\Gamma^2} \left\{ \sum_{\substack{G_1,G_2 \subset [\bar{S}]: \\ G_1 \cap G_2 \neq \varnothing}} \mathbb{1}\{|a_{G_1}| \neq 0\} \mathbb{1}\{|a_{G_2}| \neq 0\} \right\} \\
\leq\ & \frac{\sigma^4 \nu_4^{4(\bar{S}-1)} \bar{a}_m^4 (B^4\bar{S}^6) m}{v_\Gamma^2} \\
=\ & o(1),
\end{aligned}$$

under the third assumed condition.

Also, we have

$$E\left\{\sum_k E(\Delta_{mk}^2 \mid \mathscr{F}_{m,k-1})\right\} = 1.$$

Therefore, by Chebyshev's inequality,

$$\sum_k E(\Delta_{mk}^2 \mid \mathscr{F}_{m,k-1}) \to 1$$

in probability.

Step 3. Check the Lyapunov condition in equation (S4). We have

$$
\begin{aligned}
\sum_{k=1}^{m} E(\Delta_{mk}^4) &= \frac{1}{v_\Gamma^2}\left\{\sum_{s,r,t,u}^{\bar S\wedge k}\sum_{\substack{(k_1\ldots k_s)\subset[k-1]\\(k_1'\ldots k_r')\subset[k-1]\\(k_1''\ldots k_t'')\subset[k-1]\\(k_1'''\ldots k_u''')\subset[k-1]}} a_{k_1\ldots k_s k}a_{k_1'\ldots k_r' k}a_{k_1''\ldots k_t'' k}a_{k_1'''\ldots k_u''' k}E(\tilde Z_{k_1\ldots k_s}\tilde Z_{k_1'\ldots k_r'}\tilde Z_{k_1''\ldots k_t''}\tilde Z_{k_1'''\ldots k_u'''})\right\}\\
&\le \frac{\nu_4^{4\bar S}}{v_\Gamma^2}\left(\sum_{s,r,t,u}^{\bar S\wedge k}\sum_{\substack{(k_1\ldots k_s)\subset[k-1]\\(k_1'\ldots k_r')\subset[k-1]\\(k_1''\ldots k_t'')\subset[k-1]\\(k_1'''\ldots k_u''')\subset[k-1]}} |a_{k_1\ldots k_s k}||a_{k_1'\ldots k_r' k}||a_{k_1''\ldots k_t'' k}||a_{k_1'''\ldots k_u''' k}|\right)\\
&\le \frac{\nu_4^{4\bar S}\bar a^4}{v_\Gamma^2}\left(\sum_{\substack{G_1,G_2,G_3,G_4\subset[\bar S]:\\ G_1\cap G_2\cap G_3\cap G_4\neq\varnothing}} \mathbb{1}\{|a_{G_1}|\neq 0\}\mathbb{1}\{|a_{G_2}|\neq 0\}\mathbb{1}\{|a_{G_3}|\neq 0\}\mathbb{1}\{|a_{G_4}|\neq 0\}\right)\\
&\le \frac{\nu_4^{4\bar S}\bar a_m^4(B\bar S)^4 m}{v_\Gamma^2} \;=\; o(1).
\end{aligned}
$$

Combining results in Steps 1–3 and Proposition S1, we prove the results in Theorem S1. □

S2 Proofs

S2.1 Lemmas

We first introduce two Lemmas to simplify the proof.

Lemma S1. *For any two arrays $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, we have*

$$
n^{-2}\sum_{i,j} a_i b_j (p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1) \quad\le\quad n^{-1}(p^{-\bar S}-1)(\max_i a_i)(\max_i b_i)\bar S\bar D.
$$

Proof of Lemma S1. $p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1$ is nonzero if and only if $\mathcal{S}_i\cap\mathcal{S}_j\neq\varnothing$. For each unit i, the number of groups i belongs to is no larger than $\bar S$, and there are at most $\bar D$ units in each group. Therefore, for each i,

$|\sum_j b_j(p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1)| \le (p^{-\bar{S}}-1)(\max_i b_i)\bar{S}\bar{D}$ thus

$$|\sum_{i,j} a_i b_j (p^{-|S_i\cap S_j|}-1)| \le n(p^{-\bar{S}}-1)(\max_i a_i)(\max_i b_i)\bar{S}\bar{D}.$$

□

Lemma S2. *For any two arrays $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, we have*

$$\left| E\left\{ n^{-2}\sum_{i,j}\frac{T_iT_ja_ib_j(p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1)}{p^{|\mathcal{S}_i\cup\mathcal{S}_j|}}\right\}\right| \le n^{-1}(p^{-\bar{S}}-1)(\max_i a_i)(\max_i b_i)\bar{S}\bar{D}, \tag{S6}$$

$$\operatorname{var}\left\{ n^{-2}\sum_{i,j}\frac{T_iT_ja_ib_j(p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1)}{p^{|\mathcal{S}_i\cup\mathcal{S}_j|}}\right\} \le n^{-3}p^{-4\bar{S}}(\max_i a_i^2)(\max_i b_i^2)\bar{S}^3\bar{D}^3(p^{-\bar{S}}-1)^2. \tag{S7}$$

Proof of Lemma S2. By the fact that $E(T_iT_j)=p^{|\mathcal{S}_i\cup\mathcal{S}_j|}$, we have

$$E\left\{ n^{-2}\sum_{i,j}\frac{T_iT_ja_ib_j(p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1)}{p^{|\mathcal{S}_i\cup\mathcal{S}_j|}}\right\} = n^{-2}\sum_{i,j} a_ib_j(p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1).$$

Equation (S6) holds by Lemma S1.

Next, we prove equation (S7). Recall that we denote $(\Lambda_1)_{i,j}=p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1$. We have

$$\begin{aligned}\operatorname{var}\left\{ n^{-2}\sum_{i,j}\frac{T_iT_ja_ib_j(p^{-|\mathcal{S}_i\cap\mathcal{S}_j|}-1)}{p^{|\mathcal{S}_i\cup\mathcal{S}_j|}}\right\} &= n^{-4}\sum_{i,j,u,v}\frac{\operatorname{cov}(T_iT_j,T_uT_v)a_ib_ja_ub_v(\Lambda_1)_{i,j}(\Lambda_1)_{u,v}}{p^{|\mathcal{S}_i\cup\mathcal{S}_j|+|\mathcal{S}_u\cup\mathcal{S}_v|}} \\ &\le n^{-4}p^{-4\bar{S}}\sum_{i,j,u,v}|\operatorname{cov}(T_iT_j,T_uT_v)a_ib_ja_ub_v(\Lambda_1)_{i,j}(\Lambda_1)_{u,v}|. \quad \text{(S8)} \\ & \quad \text{(S9)}\end{aligned}$$

If $(\mathcal{S}_i\cup\mathcal{S}_j)\cap(\mathcal{S}_u\cup\mathcal{S}_v)=\varnothing$, then T_iT_j and T_uT_v are independent, thus $\operatorname{cov}(T_iT_j,T_uT_v)=0$. Therefore, $\operatorname{cov}(T_iT_j,T_uT_v)(\Lambda_1)_{i,j}(\Lambda_1)_{u,v}$ is nonzero if and only if $\mathcal{S}_i\cup\mathcal{S}_j\ne\varnothing$, $\mathcal{S}_u\cup\mathcal{S}_v\ne\varnothing$, and $(\mathcal{S}_i\cup\mathcal{S}_j)\cap(\mathcal{S}_u\cup\mathcal{S}_v)\ne\varnothing$. Without loss of generality, assume that $(\mathcal{S}_i\cup\mathcal{S}_j)\cap\mathcal{S}_u\ne\varnothing$, we have

$$\begin{aligned}&\sum_{i,j,u,v}|\operatorname{cov}(T_iT_j,T_uT_v)a_ib_ja_ub_v(\Lambda_1)_{i,j}(\Lambda_1)_{u,v}| \\ \le\ &\sum_{i,j}|a_ib_j|(\Lambda_1)_{i,j}\sum_{u,v}|\operatorname{cov}(T_iT_j,T_uT_v)a_ub_v(\Lambda_1)_{u,v}|\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i,j} |a_i b_j| (\Lambda_1)_{i,j} \sum_u |a_u| \sum_v |\mathrm{cov}(T_i T_j, T_u T_v) b_v (\Lambda_1)_{u,v}| \\
&\le \sum_{i,j} |a_i b_j| (\Lambda_1)_{i,j} \sum_u |a_u| (p^{-\bar{S}} - 1)(\max_v b_v) \bar{S}\bar{D} 1\{\mathcal{S}_i \cup \mathcal{S}_j) \cap \mathcal{S}_u \neq \varnothing\} \\
&\le \sum_{i,j} |a_i b_j| (\Lambda_1)_{i,j} (p^{-\bar{S}} - 1)(\max_u a_u)(\max_v b_v) \bar{S}^2 \bar{D}^2 \\
&\le n(p^{-\bar{S}} - 1)^2 (\max_u a_u^2)(\max_v b_v^2) \bar{S}^3 \bar{D}^3,
\end{aligned}
$$

where the third inequality follows from $(\mathcal{S}_i \cup \mathcal{S}_j) \cap \mathcal{S}_u \neq \varnothing$ and a similar argument in the proof of Lemma S1 that for each u, $|\sum_v b_v (\Lambda_1)_{u,v}| \le (p^{-\bar{S}} - 1)(\max_v b_v)\bar{S}\bar{D}$, the forth inequality follows from the fact that u has to be connected to either i of j, and the total number of u such that $1\{(\mathcal{S}_i \cup \mathcal{S}_j) \cap \mathcal{S}_u \neq \varnothing\}$ is nonzero is no larger than $\bar{S}\bar{D}$, and the last equality follows from Lemma S1. Plugging in back to equation (S8) gives the second inequality in Lemma S2. □

S2.2 Proof of Theorem 1

We first show $\hat{\mu}_1$ is consistent to μ_1 by showing the numerator of $\hat{\mu}_1 - \mu_1$ converges in probability to 0 and the denominator converges in probability to 1. The numerator of $\hat{\mu}_1 - \mu_1$ has mean zero and variance equal to

$$
\begin{aligned}
& \mathrm{var}\left\{ n^{-1} \sum_{i=1}^n \frac{T_i (Y_i - \mu_1)}{p^{|\mathcal{S}_i|}} \right\} \\
= \;& E\left(\left[n^{-1} \sum_{i=1}^n \frac{\{Y_i(\mathbf{1}) - \mu_1\} \mathbb{1}\{\sum_{k=1}^m W_{ik}(1 - Z_k) = 0\}}{p^{|\mathcal{S}_i|}} \right]^2 \right) \\
= \;& n^{-2} \sum_{i,j} \frac{1}{p^{|\mathcal{S}_i| + |\mathcal{S}_j|}} E\left[\{Y_i(\mathbf{1}) - \mu_1\}\{Y_j(\mathbf{1}) - \mu_1\} \mathbb{1}\left\{ \sum_{k=1}^m W_{ik}(1 - Z_k) = 0 \right\} \mathbb{1}\left\{ \sum_{k=1}^m W_{jk}(1 - Z_k) = 0 \right\} \right] \\
= \;& n^{-2} \sum_{i,j} \frac{\{Y_i(\mathbf{1}) - \mu_1\}\{Y_j(\mathbf{1}) - \mu_1\} p^{|\mathcal{S}_i \cup \mathcal{S}_j|}}{p^{|\mathcal{S}_i| + |\mathcal{S}_j|}} \\
= \;& n^{-2} \sum_{i,j} \{Y_i(\mathbf{1}) - \mu_1\}\{Y_j(\mathbf{1}) - \mu_1\} p^{-|\mathcal{S}_i \cap \mathcal{S}_j|} \\
= \;& n^{-2} \sum_{i,j} \{Y_i(\mathbf{1}) - \mu_1\}\{Y_j(\mathbf{1}) - \mu_1\} (p^{-|\mathcal{S}_i \cap \mathcal{S}_j|} - 1) \\
\le \;& n^{-1} (p^{-\bar{S}} - 1)(\max_i a_i)(\max_i b_i) \bar{S}\bar{D},
\end{aligned}
$$

where the second-to-last equality follows from the fact that $\sum_{i,j}\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{1})-\mu_1\}=0$, and the last inequality follows from Lemma S1. Therefore, the numerator of $\hat{\mu}_1-\mu_1$ converges in probability to 0 by Chebyshev's inequality. Similarly, the denominator of $\hat{\mu}_1-\mu_1$ has mean 1 and variance converging in probability to 0. This concludes the proof of $\hat{\mu}_1$ converges in probability to μ_1. Analogously, $\hat{\mu}_0$ converges in probability to μ_0, which concludes the proof of Theorem 1.

S2.3 Proof of Theorem 2

We first compute the asymptotic variance of the proposed estimators $\hat{\mu}_1$. The denominator of $\hat{\mu}_1$ converges in probability to 1. By Slutsky's Theorem, we have

$$\begin{aligned}
\text{avar}(\hat{\mu}_1) &= \text{var}\left[n^{-1}\sum_{i=1}^{n}\frac{\{Y_i(\mathbf{1})-\mu_1\}\mathbb{1}\{\sum_{k=1}^{m}W_{ik}(1-Z_k)=0\}}{p^{|\mathcal{S}_i|}}\right] \\
&= E\left(\left[n^{-1}\sum_{i=1}^{n}\frac{\{Y_i(\mathbf{1})-\mu_1\}\mathbb{1}\{\sum_{k=1}^{m}W_{ik}(1-Z_k)=0\}}{p^{|\mathcal{S}_i|}}\right]^2\right) \\
&= n^{-2}\sum_{i,j}\frac{1}{p^{|\mathcal{S}_i|+|\mathcal{S}_j|}}E\left[\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{1})-\mu_1\}\mathbb{1}\left\{\sum_{k=1}^{m}W_{ik}(1-Z_k)=0\right\}\mathbb{1}\left\{\sum_{k=1}^{m}W_{jk}(1-Z_k)=0\right\}\right] \\
&= n^{-2}\sum_{i,j}\frac{\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{1})-\mu_1\}p^{|\mathcal{S}_i\cup\mathcal{S}_j|}}{p^{|\mathcal{S}_i|+|\mathcal{S}_j|}} \\
&= n^{-2}\sum_{i,j}\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{1})-\mu_1\}p^{-|\mathcal{S}_i\cap\mathcal{S}_j|} \\
&= n^{-2}\sum_{i,j}\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{1})-\mu_1\}(\Lambda_1)_{i,j}.
\end{aligned}$$

By symmetry, we have

$$\text{avar}(\hat{\mu}_0) = n^{-2}\sum_{i,j}\{Y_i(\mathbf{0})-\mu_0\}\{Y_j(\mathbf{0})-\mu_0\}(\Lambda_0)_{i,j}.$$

Next, we compute the asymptotic covariance between $\hat{\mu}_1$ and $\hat{\mu}_0$:

$$\begin{aligned}
\text{acov}(\hat{\mu}_1,\hat{\mu}_0) &= n^{-2}E\left[\sum_{i=1}^{n}\frac{\{Y_i(\mathbf{1})-\mu_1\}\mathbb{1}\{\sum_{k=1}^{m}W_{ik}(1-Z_k)=0\}}{p^{|\mathcal{S}_i|}},\sum_{i=1}^{n}\frac{\{Y_i(\mathbf{0})-\mu_0\}\mathbb{1}\{\sum_{k=1}^{m}W_{ik}Z_k=0\}}{(1-p)^{|\mathcal{S}_i|}}\right] \\
&= n^{-2}\sum_{i,j}E\left[\frac{\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{0})-\mu_0\}\mathbb{1}\{\sum_{k=1}^{m}W_{ik}(1-Z_k)=0\}\mathbb{1}\{\sum_{k=1}^{m}W_{jk}Z_k=0\}}{p^{|\mathcal{S}_i|}(1-p)^{|\mathcal{S}_j|}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= n^{-2}\sum_{i,j}\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{0})-\mu_0\}\mathbb{1}\{S_i\cap S_j=\varnothing\}\\
&= -n^{-2}\sum_{i,j}\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{0})-\mu_0\}\mathbb{1}\{S_i\cap S_j\neq\varnothing\}.
\end{aligned}
$$

Combining the results, we have

$$
\begin{aligned}
\operatorname{avar}(\hat{\tau}) &= \operatorname{avar}(\hat{\mu}_1)+\operatorname{avar}(\hat{\mu}_2)-2\operatorname{acov}(\hat{\mu}_1,\hat{\mu}_2)\\
&= n^{-2}\sum_{i,j}\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{1})-\mu_1\}(\Lambda_1)_{i,j}+n^{-2}\sum_{i,j}\{Y_i(\mathbf{0})-\mu_0\}\{Y_j(\mathbf{0})-\mu_0\}(\Lambda_0)_{i,j}\\
&\quad +2n^{-2}\sum_{i,j}\{Y_i(\mathbf{1})-\mu_1\}\{Y_j(\mathbf{0})-\mu_0\}(\Lambda_\tau)_{i,j}\\
&= n^{-2}\left\{\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1})+\tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0})+2\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0})\right\}.
\end{aligned}
$$

We then apply the general Central Limit Theorem S1 following two steps.

Step 1. We first give an alternative representation of the numerator of $\hat{\mu}_1-\mu_1$, which is equal to

$$
\begin{aligned}
& n^{-1}\sum_{i=1}^{n}\frac{\mathbb{1}\{\sum_{k=1}^{m}W_{ik}(1-Z_{ik})=0\}\{Y_i(\mathbf{1})-\mu_1\}}{p^{|\mathcal{S}_i|}}\\
=\; & \sum_{k_1}\frac{Z_{k_1}}{np}\sum_{i=1}^{n}\mathbb{1}\{|\mathcal{S}_i|=1\}W_{ik_1}\{Y_i(\mathbf{1})-\mu_1\}\\
& +\sum_{k_1<k_2}\frac{Z_{k_1}Z_{k_2}}{np^2}\sum_{i=1}^{n}\mathbb{1}\{|\mathcal{S}_i|=2\}W_{ik_1}W_{ik_2}\{Y_i(\mathbf{1})-\mu_1\}\\
& +\cdots\\
& +\sum_{k_1<\cdots<k_{\bar{S}}}\frac{Z_{k_1}\cdots Z_{k_{\bar{S}}}}{np^{\bar{S}}}\sum_{i=1}^{n}\mathbb{1}\{|\mathcal{S}_i|=\bar{S}\}W_{ik_1}\cdots W_{ik_{\bar{S}}}\{Y_i(\mathbf{1})-\mu_1\}\\
=\; & \sum_{k_1}\frac{\tilde{Z}_{k_1}+p}{np}\sum_{i=1}^{n}\mathbb{1}\{|\mathcal{S}_i|=1\}W_{ik_1}\{Y_i(\mathbf{1})-\mu_1\}\\
& +\sum_{k_1<k_2}\frac{(\tilde{Z}_{k_1}+p)(\tilde{Z}_{k_2}+p)}{np^2}\sum_{i=1}^{n}\mathbb{1}\{|\mathcal{S}_i|=2\}W_{ik_1}W_{ik_2}\{Y_i(\mathbf{1})-\mu_1\}\\
& +\cdots\\
& +\sum_{k_1<\cdots<k_{\bar{S}}}\frac{(\tilde{Z}_{k_1}+p)\cdots(\tilde{Z}_{k_{\bar{S}}}+p)}{np^{\bar{S}}}\sum_{i=1}^{n}\mathbb{1}\{|\mathcal{S}_i|=\bar{S}\}W_{ik_1}\cdots W_{ik_{\bar{S}}}\{Y_i(\mathbf{1})-\mu_1\}. \qquad \text{(S10)}
\end{aligned}
$$

By binomial expansion, for any $s \in [\bar{S}]$, we have

$$
\begin{aligned}
& \sum_{k_1 < \cdots < k_s} \frac{(\tilde{Z}_{k_1} + p) \cdots (\tilde{Z}_{k_s} + p)}{np^s} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_s} \{Y_i(\mathbf{1}) - \mu_1\} \\
= \; & \frac{1}{s!} \sum_{k_1 \neq \ldots \neq k_s} \frac{(\tilde{Z}_{k_1} + p) \cdots (\tilde{Z}_{k_s} + p)}{np^s} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_s} \{Y_i(\mathbf{1}) - \mu_1\} \\
= \; & \frac{1}{s!} \binom{s}{1} \sum_{k_1} \frac{\tilde{Z}_{k_1}}{np} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{\substack{k_2 \neq \cdots \neq k_s, \\ k_u \neq k_1, \forall 1 < u \leq s}} W_{ik_2} \cdots W_{ik_s} \\
& + \cdots \\
& + \frac{1}{s!} \binom{s}{\ell} \sum_{k_1 \neq \ldots \neq k_\ell} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}}{np^\ell} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{\substack{k_{\ell+1} \neq \cdots \neq k_s, \\ k_u \neq k_1, \ldots, k_\ell, \forall \ell < u \leq s}} W_{ik_{\ell+1}} \cdots W_{ik_s} \\
& + \cdots \\
& + \frac{1}{s!} \binom{s}{s-1} \sum_{k_1 \neq \ldots \neq k_{s-1}} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_{s-1}}}{np^{s-1}} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_{s-1}} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{\substack{k_s, \\ k_s \neq k_1, \ldots, k_{s-1}}} W_{ik_s} \\
& + \frac{1}{s!} \sum_{k_1 \neq \ldots \neq k_s} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_s}}{np^s} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_s} \{Y_i(\mathbf{1}) - \mu_1\}. \qquad \text{(S11)}
\end{aligned}
$$

For the ℓ-th term, we have

$$
\begin{aligned}
& \frac{1}{s!} \binom{s}{\ell} \sum_{k_1 \neq \ldots \neq k_\ell} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}}{np^\ell} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{\substack{k_{\ell+1} \neq \cdots \neq k_s, \\ k_u \neq k_1, \ldots, k_\ell, \forall \ell < u \leq s}} W_{ik_{\ell+1}} \cdots W_{ik_s} \\
= \; & \frac{1}{s!} \binom{s}{\ell} \sum_{k_1 \neq \ldots \neq k_\ell} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}}{np^\ell} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{1}) - \mu_1\} (s - \ell)! \\
= \; & \frac{1}{s!} \binom{s}{\ell} (s - \ell)! \ell! \sum_{k_1 < \cdots < k_\ell} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}}{np^\ell} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{1}) - \mu_1\} \\
= \; & \sum_{k_1 < \cdots < k_\ell} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}}{np^\ell} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{1}) - \mu_1\}.
\end{aligned}
$$

Plugging back to equation (S11), we have

$$
\sum_{k_1 < \cdots < k_s} \frac{(\tilde{Z}_{k_1} + p) \cdots (\tilde{Z}_{k_s} + p)}{np^s} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_s} \{Y_i(\mathbf{1}) - \mu_1\}
$$

$$= \sum_{\ell=1}^{s} \sum_{k_1 < \cdots < k_\ell} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}}{np^\ell} \sum_{i=1}^{n} \mathbb{1}\{|\mathcal{S}_i| = s\} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{1}) - \mu_1\},$$

thus, the summation in (S10) is equal to

$$\begin{aligned}
&\sum_{k_1} \frac{\tilde{Z}_{k_1}}{np} \sum_{i=1}^{n} W_{ik_1} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{s=1}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\} \\
&+ \sum_{k_1 < k_2} \frac{\tilde{Z}_{k_1} \tilde{Z}_{k_2}}{np^2} \sum_{i=1}^{n} W_{ik_1} W_{ik_2} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{s=2}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\} \\
&+ \cdots \\
&+ \sum_{k_1 < \cdots < k_{\bar{S}}} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_{\bar{S}}}}{np^{\bar{S}}} \sum_{i=1}^{n} W_{ik_1} \cdots W_{ik_{\bar{S}}} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{s=\bar{S}}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\}.
\end{aligned}$$

By symmetry, the numerator of $\hat{\mu}_0 - \mu_0$ equals

$$\begin{aligned}
&n^{-1} \sum_{i=1}^{n} \frac{\mathbb{1}\{\sum_{k=1}^{m} W_{ik} Z_{ik} = 0\}\{Y_i(\mathbf{0}) - \mu_0\}}{n(1-p)^{|\mathcal{S}_i|}} \\
= &\sum_{k_1} - \frac{\tilde{Z}_{k_1}}{n(1-p)} \sum_{i=1}^{n} W_{ik_1} \{Y_i(\mathbf{0}) - \mu_0\} \sum_{s=1}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\} \\
&+ \sum_{k_1 < k_2} (-1)^2 \frac{\tilde{Z}_{k_1} \tilde{Z}_{k_2}}{n(1-p)^2} \sum_{i=1}^{n} W_{ik_1} W_{ik_2} \{Y_i(\mathbf{0}) - \mu_0\} \sum_{s=2}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\} \\
&+ \cdots \\
&+ \sum_{k_1 < \cdots < k_{\bar{S}}} (-1)^{\bar{S}} \frac{\tilde{Z}_{k_1} \cdots \tilde{Z}_{k_{\bar{S}}}}{n(1-p)^{\bar{S}}} \sum_{i=1}^{n} W_{ik_1} \cdots W_{ik_{\bar{S}}} \{Y_i(\mathbf{0}) - \mu_0\} \sum_{s=\bar{S}}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\}.
\end{aligned}$$

Define

$$\begin{aligned}
a_{1,k_1 \cdots k_\ell} &= \frac{1}{np^\ell} \sum_{i=1}^{n} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{1}) - \mu_1\} \sum_{s=\ell}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\}, \\
a_{0,k_1 \cdots k_\ell} &= \frac{(-1)^\ell}{n(1-p)^\ell} \sum_{i=1}^{n} W_{ik_1} \cdots W_{ik_\ell} \{Y_i(\mathbf{0}) - \mu_0\} \sum_{s=\ell}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\}.
\end{aligned}$$

To summarize, we have shown that the numerator of $\hat{\mu}_z - \mu_z$ is equal to

$$\sum_{\ell=1}^{\bar{S}} \sum_{k_1 < \cdots < k_\ell} a_{z,k_1 \cdots k_\ell} \tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}$$

for $z = 1, 0$.

Step 2. We now consider any linear combination of the numerators of $\hat{\mu}_1 - \mu_1$ and $\hat{\mu}_0 - \mu_0$. We show it can be reformulated in the form of Γ defined in equation (S1). Consider any c_1 and c_0 that has $c_1^2 + c_0^2 = 1$. Define

$$a_{k_1 \cdots k_\ell} \quad = \quad c_1 a_{1,k_1 \cdots k_\ell} + c_0 a_{0,k_1 \cdots k_\ell}.$$

Then we can write

$$n^{-1} \sum_{i=1}^{n} \left[\frac{c_1 T_i \{Y_i(\mathbf{1}) - \mu_1\}}{p^{|\mathcal{S}_i|}} + \frac{c_0 C_i \{Y_i(\mathbf{0}) - \mu_0\}}{(1-p)^{|\mathcal{S}_i|}} \right] \quad = \quad \sum_{\ell=1}^{\bar{S}} \sum_{k_1 < \cdots < k_\ell} a_{k_1 \cdots k_\ell} \tilde{Z}_{k_1} \cdots \tilde{Z}_{k_\ell}. \tag{S12}$$

We will apply Theorem S1 to establish a CLT for (S12). We check the two conditions required in Theorem S1.

We first show the boundedness of a's. Note that

$$a_{k_1 \cdots k_\ell} \quad = \quad \sum_{i=1}^{n} W_{ik_1} \cdots W_{ik_\ell} \left[\frac{c_1 \{Y_i(\mathbf{1}) - \mu_1\}}{np^\ell} + \frac{(-1)^\ell c_0 \{Y_i(\mathbf{0}) - \mu_0\}}{n(1-p)^\ell} \right] \sum_{s=\ell}^{\bar{S}} \mathbb{1}\{|\mathcal{S}_i| = s\}.$$

The summand indexed by i is nonzero only if unit i belongs to groups $k_1, \ldots, k_\ell$. By Assumption 2, for each $k_1, \ldots, k_\ell$, we have at most $\bar{D}$ such units. Hence we obtain

$$|a_{k_1 \cdots k_\ell}| \quad \leq \quad \frac{\bar{D} \max_i \{|Y_i(\mathbf{1}) - \mu_1|, |Y_i(\mathbf{0}) - \mu_0|\}}{n} \left\{ p^{-\bar{S}} + (1-p)^{-\bar{S}} \right\} := \bar{a}_m.$$

Second, we verify the limited overlapping condition $\sum_{(k_1 \cdots k_s) \subset [m] \backslash \{k\}} \mathbb{1}\{|a_{kk_1 \ldots k_s}| \neq 0\} \leq B$. For any $(k_1 \cdots k_s) \subset [m]$ and k, we have $\mathbb{1}\{|a_{kk_1 \ldots k_s}| \neq 0\} = \mathbb{1}\{\exists\ i$, such that $W_{ik_1} \cdots W_{ik_s} W_{ik} = 1\}$, which is nonzero if and only if $k_1, \ldots, k_s$ are all connected to group k. Therefore,

$$\sum_{(k_1 \cdots k_s) \subset [m] \backslash \{k\}} \mathbb{1}\{|a_{kk_1 \ldots k_s}| \neq 0\} \quad \leq \quad \sum_{(k_1 \cdots k_s) \subset [m] \backslash \{k\}} \mathbb{1}\{k_1, \ldots, k_s \text{ are all connected to group } k\} \ \leq\ B^s,$$

where the last inequality holds because by Assumption 4, there are at most B groups connected to group k, thus the number of combinations $(k_1, \ldots, k_s)$ such that all of them are connected to k is upper bounded by

$\binom{B}{s} \leq B^s$.

Therefore, by Step 2, we conclude that the numerators of $\hat{\mu}_1 - \mu_1$ and $\hat{\mu}_0 - \mu_0$ converge jointly to a bivariate standard normal distribution, after standardization via

$$\begin{pmatrix} n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}) & n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{1})\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0}) \\ n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{1})\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0}) & n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}) \end{pmatrix}.$$

Moreover, the denominators of $\hat{\mu}_1$ and $\hat{\mu}_0$ are converging in probability to 1, thus the asymptotic distribution in Theorem 2 holds by Slutsky's Theorem.

□

S2.4 Proof of Theorem 3

We first prove the convergence of $\hat{v}/\mathrm{plim}(\hat{v})$. Denote

$$\begin{aligned} \hat{v}_1 &= n^{-2}\sum_{i,j}\frac{T_iT_j(Y_i-\hat{\mu}_1)(Y_j-\hat{\mu}_1)(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}}, \\ \hat{v}_0 &= n^{-2}\sum_{i,j}\frac{C_iC_j(Y_i-\hat{\mu}_0)(Y_j-\hat{\mu}_0)(\Lambda_0)_{i,j}}{(1-p)^{|S_i\cup S_j|}}, \end{aligned}$$

then we have $\hat{v} = (\hat{v}_1^{1/2} + \hat{v}_0^{1/2})^2$. We prove the convergence of $\hat{v}/\mathrm{plim}(\hat{v})$ by showing that $\hat{v}_1/\mathrm{plim}(\hat{v}_1) = 1 + o_p(1)$ and $\hat{v}_0/\mathrm{plim}(\hat{v}_0) = 1 + o_p(1)$, where

$$\begin{aligned} \mathrm{plim}(\hat{v}_1) &= \mathrm{avar}(\hat{\mu}_1) = n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}), \\ \mathrm{plim}(\hat{v}_0) &= \mathrm{avar}(\hat{\mu}_0) = n^{-2}\tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}). \end{aligned}$$

Rewrite

$$\begin{aligned} \hat{v}_1 &= n^{-2}\sum_{i,j}\frac{T_iT_j\{Y_i-\mu_1+(\mu_1-\hat{\mu}_1)\}\{Y_j-\mu_1+(\mu_1-\hat{\mu}_1)\}(p^{-|S_i\cap S_j|}-1)}{p^{|S_i\cup S_j|}} \\ &= n^{-2}\sum_{i,j}\frac{T_iT_j(Y_i-\mu_1)(Y_j-\mu_1)(p^{-|S_i\cap S_j|}-1)}{p^{|S_i\cup S_j|}} && \text{(S13)} \\ &\quad +2(\mu_1-\hat{\mu}_1)n^{-2}\sum_{i,j}\frac{T_iT_j(Y_i-\mu_1)(p^{-|S_i\cap S_j|}-1)}{p^{|S_i\cup S_j|}} && \text{(S14)} \\ &\quad +(\mu_1-\hat{\mu}_1)^2n^{-2}\sum_{i,j}\frac{T_iT_j(p^{-|S_i\cap S_j|}-1)}{p^{|S_i\cup S_j|}}, && \text{(S15)} \end{aligned}$$

and use $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ to denote the three terms in (S13)–(S15), respectively. By the fact that $E(T_iT_j) = p^{|S_i \cup S_j|}$, we have $E(\mathcal{T}_1) = \text{plim}(\hat{v}_1)$. The variance of $\mathcal{T}_1$,

$$\text{var}(\mathcal{T}_1) \quad \leq \quad n^{-3}p^{-4\bar{S}}[\max_i \{Y_i(\mathbf{1}) - \mu_1\}^4]\bar{S}^3\bar{D}^3(p^{-\bar{S}} - 1)^2 \;=\; O_p(n^{-3}\bar{D}^3)$$

by Lemma S2 when taking $a_i = b_i = Y_i(\mathbf{1}) - \mu_1$. Thus,

$$\mathcal{T}_1 \quad = \quad E(\mathcal{T}_1) + O_p\{\text{var}(\mathcal{T}_1)^{1/2}\} \;=\; \text{plim}(\hat{v}_1) + O_p(n^{-3/2}\bar{D}^{3/2}) \;=\; \text{plim}(\hat{v}_1) + o_p(n^{-1}\bar{D}).$$

Similarly, by Lemma S2, we have

$$E\left\{n^{-2}\sum_{i,j}\frac{T_iT_j(Y_i - \mu_1)(p^{-|S_i \cap S_j|} - 1)}{p^{|S_i \cup S_j|}}\right\} \quad = \quad O_p(n^{-1}\bar{D}),$$
$$\text{var}\left\{n^{-2}\sum_{i,j}\frac{T_iT_j(Y_i - \mu_1)(p^{-|S_i \cap S_j|} - 1)}{p^{|S_i \cup S_j|}}\right\} \quad = \quad O_p(n^{-3}\bar{D}^3),$$

by taking $a_i = Y_i(\mathbf{1}) - \mu_1$ and $b_i = 1$. By the proof of Theorem 2, we have $\hat{\mu}_1 - \mu_1 = O_p(n^{-1/2}\bar{D}^{1/2})$, and $\mathcal{T}_2 = E(\mathcal{T}_2) + O_p\{\text{var}(\mathcal{T}_2)^{1/2}\}$ gives us

$$\mathcal{T}_2 \quad = \quad O_p(n^{-1/2}\bar{D}^{1/2}) \cdot O_p(n^{-1}\bar{D}) + O_p[\{n^{-1}\bar{D} \cdot n^{-3}\bar{D}^3\}^{1/2}] \;=\; o_p(n^{-1}\bar{D}).$$

Also, we have

$$E\left\{n^{-2}\sum_{i,j}\frac{T_iT_j(p^{-|S_i \cap S_j|} - 1)}{p^{|S_i \cup S_j|}}\right\} \quad = \quad O_p(n^{-1}\bar{D}),$$
$$\text{var}\left\{n^{-2}\sum_{i,j}\frac{T_iT_j(p^{-|S_i \cap S_j|} - 1)}{p^{|S_i \cup S_j|}}\right\} \quad = \quad O_p(n^{-3}\bar{D}^3)$$

by taking $(a_i, b_i) = (1, 1)$ in Lemma S2. Again, we have

$$\mathcal{T}_3 \quad = \quad O_p(n^{-1}\bar{D}) \cdot O_p(n^{-1}\bar{D}) + O_p[\{n^{-2}\bar{D}^2 \cdot n^{-3}\bar{D}^3\}^{1/2}] \;=\; o_p(n^{-1}\bar{D}).$$

Combining the three terms $\mathcal{T}_1$–$\mathcal{T}_3$, we have

$$\hat{v}_1 \quad = \quad \text{plim}(\hat{v}_1) + o_p(n^{-1}\bar{D}).$$

Under the regularity condition that the weighted covariance matrix of the potential outcomes $Y_i(\mathbf{1})$ and $Y_i(\mathbf{0})$ are non-degenerated, $\text{plim}(\hat{v}_1) = O_p(n^{-1}\bar{D})$, thus $\hat{v}_1/\text{plim}(\hat{v}_1) = 1 + o_p(1)$. Analogously, $\hat{v}_0/\text{plim}(\hat{v}_0) = 1 + o_p(1)$. By the continuous mapping theorem, $\hat{v}/\text{plim}(\hat{v})$ converges in probability to 1.

Next, we prove that $\text{plim}(\hat{v}) \geq \text{avar}(\hat{\tau})$. Recall that $\text{plim}(\hat{v}) = \{\text{plim}(\hat{v}_1)^{1/2} + \text{plim}(\hat{v}_0)^{1/2}\}^2$, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\text{avar}(\hat{\tau}) &= \text{avar}(\hat{\mu}_1) + \text{avar}(\hat{\mu}_0) - 2\text{acov}(\hat{\mu}_1, \hat{\mu}_0) \\ &\leq \text{plim}(\hat{v}_1) + \text{plim}(\hat{v}_0) + 2\text{plim}(\hat{v}_1)^{1/2}\text{plim}(\hat{v}_0)^{1/2} \;=\; v.\end{aligned}$$

□

S2.5 Proof of Proposition 1

The proof for the consistency and asymptotic normality of $\hat{\tau}(\beta_1, \beta_0)$ is analogous to that of Theorem 1 and 2. We only need to treat the $Y_i(1) - \tilde{X}_i^{\mathrm{T}}\beta_1$ and $Y_i(0) - \tilde{X}_i^{\mathrm{T}}\beta_0$ as pseudo potential outcomes. The remaining step is to check that the conditions in Theorems 1 and 2 still hold with the pseudo potential outcomes. Assumptions 1, 2 and 4 still hold because the network structure remains the same. To check Assumption 3, suppose $|Y_i(z)| \leq a_Y$ and $|X_{ik}| \leq a_X$, then we have $|Y_i - \beta^{\mathrm{T}}\tilde{X}_i| \leq a_Y + \|\beta\|_1 a_X$ is also bounded. □

S2.6 Proof of Proposition 2

The result follows from the fact that

$$\begin{aligned}n^2 v_n(\tilde{\beta}_1, \tilde{\beta}_0) &= \{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\tilde{\beta}_1\}^{\mathrm{T}}\Lambda_1\{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\tilde{\beta}_1\} + \{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\tilde{\beta}_0\}^{\mathrm{T}}\Lambda_0\{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\tilde{\beta}_0\} \\ &\quad + 2\{\tilde{\boldsymbol{Y}}(\mathbf{1}) - \tilde{\boldsymbol{X}}\tilde{\beta}_1\}^{\mathrm{T}}\Lambda_\tau\{\tilde{\boldsymbol{Y}}(\mathbf{0}) - \tilde{\boldsymbol{X}}\tilde{\beta}_0\} \\ &= \tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}) + \tilde{\boldsymbol{Y}}(\mathbf{0})^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}) + 2\tilde{\boldsymbol{Y}}(\mathbf{1})^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0}) \\ &\quad + \tilde{\beta}_1^{\mathrm{T}}\tilde{X}^{\mathrm{T}}\Lambda_1\tilde{X}\tilde{\beta}_1 + \tilde{\beta}_0^{\mathrm{T}}\tilde{X}^{\mathrm{T}}\Lambda_0\tilde{X}\tilde{\beta}_0 + 2\tilde{\beta}_1^{\mathrm{T}}\tilde{X}^{\mathrm{T}}\Lambda_\tau\tilde{X}\tilde{\beta}_0 \\ &\quad - \tilde{\beta}_1^{\mathrm{T}}\tilde{X}^{\mathrm{T}}\{\Lambda_1\tilde{\boldsymbol{Y}}(\mathbf{1}) + \Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{0})\} + \tilde{\beta}_0^{\mathrm{T}}\tilde{X}^{\mathrm{T}}\{\Lambda_0\tilde{\boldsymbol{Y}}(\mathbf{0}) + \Lambda_\tau\tilde{\boldsymbol{Y}}(\mathbf{1})\} \\ &= n^2\{v_n + L(\tilde{\beta}_1, \tilde{\beta}_0)\} \;\leq\; n^2 v_n\end{aligned}$$

where the last inequality follows from the constraint in (6). □

S2.7 Proof of Theorem 4

Convergence of the regression coefficients. Define the population limit counterpart for the closed-form solution $(\tilde{\beta}_1, \tilde{\beta}_0)$:

$$\begin{pmatrix} \beta_1^\star \\ \beta_0^\star \end{pmatrix} = \Omega_{xx}^{-1}\Omega_{yx}, \tag{S16}$$

where

$$\Omega_{xx} = \begin{pmatrix} \Omega_{xx,11} & \Omega_{xx,10} \\ \Omega_{xx,01} & \Omega_{xx,00} \end{pmatrix}, \quad \Omega_{yx} = \begin{pmatrix} \Omega_{yx,11} + \Omega_{yx,01} \\ \Omega_{yx,00} + \Omega_{yx,10} \end{pmatrix}. \tag{S17}$$

By Assumption 5, we have

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_0 \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_1\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}} \\ \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_\tau\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^{\mathrm{T}}\Lambda_0\tilde{\boldsymbol{X}} \end{pmatrix} \to \Omega_{xx}.$$

By similar arguments as in Theorem 3, under Assumption 5, the following holds asymptotically in probability:

$$\begin{pmatrix} \sum_{i,j} \frac{T_iT_j\tilde{X}_i(Y_j-\hat{\mu}_1)(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}} + \sum_{i,j} \frac{C_iC_j\tilde{X}_i(Y_j-\hat{\mu}_0)(\Lambda_\tau)_{i,j}}{(1-p)^{|S_i\cup S_j|}} \\ \sum_{i,j} \frac{T_iT_j\tilde{X}_i(Y_j-\hat{\mu}_1)(\Lambda_\tau)_{i,j}}{p^{|S_i\cup S_j|}} + \sum_{i,j} \frac{C_iC_j\tilde{X}_i(Y_j-\hat{\mu}_0)(\Lambda_0)_{i,j}}{(1-p)^{|S_i\cup S_j|}} \end{pmatrix}^{\mathrm{T}} \to \begin{pmatrix} \Omega_{yx,11} + \Omega_{yx,01} \\ \Omega_{yx,00} + \Omega_{yx,10} \end{pmatrix}.$$

Therefore, we conclude that

$$(\hat{\beta}_1, \hat{\beta}_0) - (\beta_1^\star, \beta_0^\star) = o_p(1). \tag{S18}$$

Consistency and asymptotic distribution of $\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0)$. The difference between $\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0)$ and $\hat{\tau}(\beta_1^\star, \beta_0^\star)$ is

$$\hat{\tau}(\hat{\beta}_1, \hat{\beta}_0) - \hat{\tau}(\beta_1^\star, \beta_0^\star) = n^{-1}\sum_{i=1}^n \frac{T_i(\hat{\beta}_1 - \beta_1^\star)^{\mathrm{T}}\tilde{X}_i}{p^{|\mathcal{S}_i|}} \Big/ n^{-1}\sum_{i=1}^n \frac{T_i}{p^{|\mathcal{S}_i|}} - n^{-1}\sum_{i=1}^n \frac{C_i(\hat{\beta}_0 - \beta_0^\star)^{\mathrm{T}}\tilde{X}_i}{(1-p)^{|\mathcal{S}_i|}} \Big/ n^{-1}\sum_{i=1}^n \frac{C_i}{(1-p)^{|\mathcal{S}_i|}}.$$

By the consistency of the optimization solutions

$$\hat{\beta}_1 - \beta_1^\star = o_p(1), \quad \hat{\beta}_0 - \beta_0^\star = o_p(1),$$

and the facts that

$$n^{-1}\sum_{i=1}^{n}\frac{T_i\tilde{X}_i}{p^{|\mathcal{S}_i|}}=O_p\left(n^{-1/2}\bar{D}^{1/2}\right),\quad n^{-1}\sum_{i=1}^{n}\frac{C_i\tilde{X}_i}{(1-p)^{|\mathcal{S}_i|}}=O_p\left(n^{-1/2}\bar{D}^{1/2}\right),$$
$$n^{-1}\sum_{i=1}^{n}\frac{T_i}{p^{|\mathcal{S}_i|}}=1+O_p\left(n^{-1/2}\bar{D}^{1/2}\right),\quad n^{-1}\sum_{i=1}^{n}\frac{C_i}{(1-p)^{|\mathcal{S}_i|}}=1+O_p\left(n^{-1/2}\bar{D}^{1/2}\right),$$

following similar arguments as in proof of Theorem 2, we can conclude that

$$|\hat{\tau}(\hat{\beta}_1,\hat{\beta}_0)-\hat{\tau}(\beta_1^{\star},\beta_0^{\star})|=o_p\left(n^{-1/2}\bar{D}^{1/2}\right).$$

By Proposition 1, $\hat{\tau}(\beta_1^{\star},\beta_0^{\star})$ converges in probability to τ. Hence $\hat{\tau}(\hat{\beta}_1,\hat{\beta}_0)$ is also consistent to τ.

The asymptotic distribution of $\hat{\tau}(\hat{\beta}_1,\hat{\beta}_0)$ follows from Slutsky's Theorem and the fact that

$$\{v_n(\beta_1^{\star},\beta_0^{\star})\}^{-1/2}\left\{\hat{\tau}(\hat{\beta}_1,\hat{\beta}_0)-\tau\right\}\quad\rightarrow\quad\mathcal{N}(0,1)$$

in distribution.

□

S2.8 Proof of Theorem 5

$$\begin{aligned}
& n^{-2}\sum_{i,j}\frac{T_iT_j(Y_i-\hat{\mu}_1-\hat{\beta}_1^{\mathrm{T}}\tilde{X}_i)(Y_j-\hat{\mu}_1-\hat{\beta}_1^{\mathrm{T}}\tilde{X}_j)(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}}\\
= \; & n^{-2}\sum_{i,j}\frac{T_iT_j\{\tilde{Y}_i-\beta_1^{\star\mathrm{T}}\tilde{X}_i-(\hat{\mu}_1-\mu_1)-(\hat{\beta}_1-\beta^{\star})^{\mathrm{T}}\tilde{X}_i\}\{\tilde{Y}_j-\beta_1^{\star\mathrm{T}}\tilde{X}_j-(\hat{\mu}_1-\mu_1)-(\hat{\beta}_1-\beta^{\star})^{\mathrm{T}}\tilde{X}_j\}(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}}\\
= \; & n^{-2}\sum_{i,j}\frac{T_iT_j(\tilde{Y}_i-\beta_1^{\star\mathrm{T}}\tilde{X}_i)(\tilde{Y}_j-\beta_1^{\star\mathrm{T}}\tilde{X}_j)(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}}\\
& +2n^{-2}\sum_{i,j}\frac{T_iT_j(\tilde{Y}_i-\beta_1^{\star\mathrm{T}}\tilde{X}_i)\{(\hat{\mu}_1-\mu_1)+(\hat{\beta}_1-\beta^{\star})^{\mathrm{T}}\tilde{X}_j\}(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}}\\
& +n^{-2}\sum_{i,j}\frac{T_iT_j\{(\hat{\mu}_1-\mu_1)+(\hat{\beta}_1-\beta^{\star})^{\mathrm{T}}\tilde{X}_i\}\{(\hat{\mu}_1-\mu_1)+(\hat{\beta}_1-\beta^{\star})^{\mathrm{T}}\tilde{X}_j\}(\Lambda_1)_{i,j}}{p^{|S_i\cup S_j|}}\\
= \; & \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

Following a similar proof as that of Theorem 3, we have

$$\mathrm{I} = v_1^\star(\beta_1^\star) + O_p(n^{-3/2}\bar{D}^{-3/2}), \quad \mathrm{II} = o_p(n^{-3/2}\bar{D}^{-3/2}), \quad \mathrm{III} = o_p(n^{-3/2}\bar{D}^{-3/2}).$$

Meanwhile, due to the fact that $v_1^\star(\beta_1) \asymp n^{-1}\bar{D}$, we have

$$\hat{v}_{1,n}(\hat{\beta}_1) = v_1^\star(\beta_1^\star) + O_p(n^{-3/2}\bar{D}^{-3/2})$$

and similarly

$$\hat{v}_{0,n}(\hat{\beta}_0) = v_0^\star(\beta_0^\star) + O_p(n^{-3/2}\bar{D}^{-3/2}).$$

Therefore,

$$\left\{\hat{v}_{1,n}(\hat{\beta}_1)\right\}^{1/2} + \left\{\hat{v}_{0,n}(\hat{\beta}_0)\right\}^{1/2} = \{v_1^\star(\beta_1^\star)\}^{1/2} + \{v_0^\star(\beta_0^\star)\}^{1/2} + O_p(n^{-3/4}\bar{D}^{-3/4}),$$

and thus

$$\frac{\{\hat{v}_{1,n}(\hat{\beta}_1)\}^{1/2} + \{\hat{v}_{0,n}(\hat{\beta}_0)\}^{1/2}}{\{v_1^\star(\beta_1^\star)\}^{1/2} + \{v_0^\star(\beta_0^\star)\}^{1/2}} = 1 + O_p(n^{-1/4}\bar{D}^{-1/4})$$

by the fact that $v_1^\star(\beta_1^\star) \asymp n^{-1}\bar{D}$ and $v_0^\star(\beta_0^\star) \asymp n^{-1}\bar{D}$. Therefore, $\hat{v}_{n,\mathrm{UB}}(\hat{\beta}_1, \hat{\beta}_0)/v_{n,\mathrm{UB}}(\beta_1^\star, \beta_0^\star)$ converges in probability to 1.

The conservativeness of $v_{\mathrm{UB}}^\star(\beta_1^\star, \beta_0^\star)$ for the true variance $v^\star(\beta_1^\star, \beta_0^\star)$ can be established similarly to Theorem 3 when no covariates are adjusted. The trick is to take $\boldsymbol{Y}(\mathbf{1}) - \tilde{\boldsymbol{X}}\beta_1^\star$ and $\boldsymbol{Y}(\mathbf{0}) - \tilde{\boldsymbol{X}}\beta_0^\star$ as pseudo potential outcomes and apply the Cauchy-Schwarz inequality for the covariance. Details are omitted.

□

S3 Auxiliary results

This section provides additional results in the special case when $\bar{S} = 2$ and we provide a tighter rate.

S3.1 $\bar{S} = 2$

Without loss of generality, assume $k \leq k'$,

$$\begin{aligned}
\Delta_k &= \tilde{Z}_k \left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right), \\
\Delta_{k'} &= \tilde{Z}_{k'} \left(a_{k'} + a_{1k'}\tilde{Z}_1 + \cdots + a_{k-1,k'}\tilde{Z}_{k-1} + \cdots + a_{k'-1,k'}\tilde{Z}_{k'-1}\right).
\end{aligned}$$

$$\begin{aligned}
\Delta_{k'}^2 &= \tilde{Z}_{k'}^2 \left\{ \left(a_{k'} + a_{1k'}\tilde{Z}_1 + \cdots + a_{k-1,k'}\tilde{Z}_{k-1}\right)^2 + \left(a_{k,k'}\tilde{Z}_k + \cdots + a_{k'-1,k'}\tilde{Z}_{k'-1}\right)^2 \right. \\
&\quad \left. +2\left(a_{k'} + a_{1k'}\tilde{Z}_1 + \cdots + a_{k-1,k'}\tilde{Z}_{k-1}\right)\left(a_{k,k'}\tilde{Z}_k + \cdots + a_{k'-1,k'}\tilde{Z}_{k'-1}\right)\right\}.
\end{aligned}$$

We can write the covariance $\text{cov}(\Delta_k^2, \Delta_{k'}^2)$ as $\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$, where

$$\begin{aligned}
\mathcal{T}_1 &= \text{cov}\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2, \tilde{Z}_{k'}^2\left(a_k' + a_{1k'}\tilde{Z}_1 + \cdots + a_{k-1,k'}\tilde{Z}_{k-1}\right)^2\right\} \\
\mathcal{T}_2 &= \text{cov}\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2, \tilde{Z}_{k'}^2\left(a_{k,k'}\tilde{Z}_k + \cdots + a_{k'-1,k'}\tilde{Z}_{k'-1}\right)^2\right\} \\
\mathcal{T}_3 &= 2\text{cov}\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2, \right. \\
&\quad \left. \tilde{Z}_{k'}^2\left(a_k' + a_{1k'}\tilde{Z}_1 + \cdots + a_{k-1,k'}\tilde{Z}_{k-1}\right)\left(a_{k,k'}\tilde{Z}_k + \cdots + a_{k'-1,k'}\tilde{Z}_{k'-1}\right)\right\}.
\end{aligned}$$

We derive the three terms separately to show that $\text{cov}(\Delta_k^2, \Delta_{k'}^2)$ is of small order. First, we have

$$\begin{aligned}
\mathcal{T}_2 &= a_{k,k'}^2\text{cov}\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2, \tilde{Z}_{k'}^2\tilde{Z}_k^2\right\} \\
&\quad +\text{cov}\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2, \left(a_{k+1,k'}\tilde{Z}_{k+1} + \cdots + a_{k'-1,k'}\tilde{Z}_{k'-1}\right)^2\right\} \\
&\quad +\text{cov}\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2, 2a_{kk'}^2\tilde{Z}_k\left(a_{k+1,k'}\tilde{Z}_{k+1} + \cdots + a_{k'-1,k'}\tilde{Z}_{k'-1}\right)\right\} \\
&= a_{k,k'}^2\text{cov}\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2, \tilde{Z}_{k'}^2\tilde{Z}_k^2\right\} + 0 + 0 \\
&= a_{k,k'}^2\left[E\left\{\tilde{Z}_k^4\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2\tilde{Z}_{k'}^2\right\} - E\left\{\tilde{Z}_k^2\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2 E(\tilde{Z}_{k'}^2\tilde{Z}_k^2)\right\}\right] \\
&= a_{k,k'}^2\left[\left\{E(\tilde{Z}_k^4)E(\tilde{Z}_{k'}^2) - E^2(\tilde{Z}_k^2)E(\tilde{Z}_{k'}^2)\right\}E\left\{\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2\right\}\right] \\
&= a_{k,k'}^2\left[\text{var}(\tilde{Z}_k^2)E(\tilde{Z}_{k'}^2)E\left\{\left(a_k + a_{1k}\tilde{Z}_1 + \cdots + a_{k-1,k}\tilde{Z}_{k-1}\right)^2\right\}\right],
\end{aligned}$$

where the second equality follows from the facts that

$$\begin{aligned}\mathrm{cov}(\tilde{Z}_k^2\tilde{Z}_{i_1}\tilde{Z}_{i_2},\tilde{Z}_{j_1}\tilde{Z}_{j_2}) &= 0 \quad \text{for any } i_1,i_2\le k \text{ and } j_1,j_2>k, \text{ and}\\ \mathrm{cov}(\tilde{Z}_k^2\tilde{Z}_{i_1}\tilde{Z}_{i_2},\tilde{Z}_k\tilde{Z}_j) &= 0 \quad \text{for any } i_1,i_2\le k \text{ and } j>k,\end{aligned}$$

and the fourth equality follows from the independence of the centered treatment assignments $\tilde{Z}_k, \tilde{Z}_{k'}$ for any (k,k') pair.

Second, we have

$$\begin{aligned}\frac{1}{2}\mathcal{T}_3 &= \mathrm{cov}\left\{\tilde{Z}_k^2\left(a_k+a_{1k}\tilde{Z}_1+\cdots+a_{k-1,k}\tilde{Z}_{k-1}\right)^2, \tilde{Z}_{k'}^2 a_{k'}\left(a_{k,k'}\tilde{Z}_k+\cdots+a_{k'-1,k'}\tilde{Z}_{k'-1}\right)\right\}\\ &\quad +\mathrm{cov}\left\{\tilde{Z}_k^2\left(a_k+a_{1k}\tilde{Z}_1+\cdots+a_{k-1,k}\tilde{Z}_{k-1}\right)^2,\right.\\ &\qquad\left.\tilde{Z}_{k'}^2\left(a_{1k'}\tilde{Z}_1+\cdots+a_{k-1,k'}\tilde{Z}_{k-1}\right)\left(a_{k,k'}\tilde{Z}_k+\cdots+a_{k'-1,k'}\tilde{Z}_{k'-1}\right)\right\}\\ &= \mathrm{cov}\left\{\tilde{Z}_k^2\left(a_k+a_{1k}\tilde{Z}_1+\cdots+a_{k-1,k}\tilde{Z}_{k-1}\right)^2, a_{k,k'}a_{k'}\tilde{Z}_k\tilde{Z}_{k'}^2\right\},\end{aligned}$$

where the second equality follows from the facts that

$$\mathrm{cov}(\tilde{Z}_k^2\tilde{Z}_{i_1}\tilde{Z}_{i_2},\tilde{Z}_{k'}^2\tilde{Z}_j) = 0 \qquad \text{for any } i_1,i_2\le k \text{ and } j\ge k.$$

Third, we have

$$\begin{aligned}\mathcal{T}_1 &= \mathrm{cov}\left\{\tilde{Z}_k^2(a_k+a_{1k}\tilde{Z}_1+\cdots+a_{k-1,k}\tilde{Z}_{k-1})^2, \tilde{Z}_{k'}^2(a_k'+a_{1k'}\tilde{Z}_1+\cdots+a_{k-1,k'}\tilde{Z}_{k-1})^2\right\}\\ &= \mathrm{cov}\left[\tilde{Z}_k^2\left\{a_k^2+2a_k\sum_{t=1}^{k-1}a_{tk}\tilde{Z}_t+\left(\sum_{t=1}^{k-1}a_{tk}\tilde{Z}_t\right)^2\right\}, \tilde{Z}_{k'}^2\left\{a_{k'}^2+2a_{k'}\sum_{t=1}^{k-1}a_{tk'}\tilde{Z}_t+\left(\sum_{t=1}^{k-1}a_{tk'}\tilde{Z}_t\right)^2\right\}\right]\\ &= \underbrace{\mathrm{cov}\left\{2a_k\tilde{Z}_k^2\sum_{t=1}^{k-1}a_{tk}\tilde{Z}_t, 2a_{k'}\tilde{Z}_{k'}^2\sum_{t=1}^{k-1}a_{tk'}\tilde{Z}_t\right\}}_{\mathcal{T}_{11}} + \underbrace{\mathrm{cov}\left\{2a_k\tilde{Z}_k^2\sum_{t=1}^{k-1}a_{tk}\tilde{Z}_t, \tilde{Z}_{k'}^2\left(\sum_{t=1}^{k-1}a_{tk'}\tilde{Z}_t\right)^2\right\}}_{\mathcal{T}_{12}}\\ &\quad \underbrace{+\mathrm{cov}\left\{\tilde{Z}_k^2\left(\sum_{t=1}^{k-1}a_{tk}\tilde{Z}_t\right)^2, 2a_{k'}\tilde{Z}_{k'}^2\sum_{t=1}^{k-1}a_{tk'}\tilde{Z}_t\right\}}_{\mathcal{T}_{13}} + \underbrace{\mathrm{cov}\left\{\tilde{Z}_k^2\left(\sum_{t=1}^{k-1}a_{tk}\tilde{Z}_t\right)^2, \tilde{Z}_{k'}^2\left(\sum_{t=1}^{k-1}a_{tk'}\tilde{Z}_t\right)^2\right\}}_{\mathcal{T}_{14}}.\end{aligned}$$

We compute the four terms $\mathcal{T}_{11}$–$\mathcal{T}_{14}$ separately.

$$
\begin{aligned}
\mathcal{T}_{11} &= \sum_{t=1}^{k-1} \operatorname{cov}\left(2a_k \tilde{Z}_k^2 a_{tk} \tilde{Z}_t, 2a_{k'} \tilde{Z}_{k'}^2 a_{tk'} \tilde{Z}_t\right) \\
&= 2a_k a_{k'} \sum_{t=1}^{k-1} a_{tk} a_{tk'} \operatorname{cov}\left(\tilde{Z}_k^2 \tilde{Z}_t, \tilde{Z}_{k'}^2 \tilde{Z}_t\right) \\
&= O(B),
\end{aligned}
$$

where the first equality follows from the fact that $\operatorname{cov}(\tilde{Z}_k^2 \tilde{Z}_i, \tilde{Z}_{k'}^2 \tilde{Z}_j) = 0$ for any $i \neq j$ and $i, j < k$; the last equality follows from the fact that for all $t \in [k-1]$, there are at most B of them are connected to both group k and k' by Assumption 4, therefore, $\sum_{t=1}^{k-1} a_{tk} a_{tk'} \operatorname{cov}(\tilde{Z}_k^2 \tilde{Z}_t, \tilde{Z}_{k'}^2 \tilde{Z}_t)$ has at most B nonzero terms by the definition of a_k and $a_{k'}$, and $\mathcal{T}_{11}$ is of order $O(B)$.

$$
\begin{aligned}
\mathcal{T}_{12} &= \operatorname{cov}\left\{2a_k \tilde{Z}_k^2 \sum_{t=1}^{k-1} a_{tk} \tilde{Z}_t, \tilde{Z}_{k'}^2 \left(\sum_{t=1}^{k-1} a_{tk'}^2 \tilde{Z}_t^2 + 2 \sum_{1 \leq t_1 < t_2 \leq k-1} a_{t_1 k'} a_{t_2 k'} \tilde{Z}_{t_1} \tilde{Z}_{t_2}\right)\right\} \\
&= \operatorname{cov}\left\{2a_k \tilde{Z}_k^2 \sum_{t=1}^{k-1} a_{tk} \tilde{Z}_t, \tilde{Z}_{k'}^2 \sum_{t=1}^{k-1} a_{tk'}^2 \tilde{Z}_t^2\right\} + 0 \\
&= 2a_k \sum_{t=1}^{k-1} a_{tk} a_{tk'}^2 \operatorname{cov}\left(\tilde{Z}_k^2 \tilde{Z}_t, \tilde{Z}_{k'}^2 \tilde{Z}_t^2\right) \\
&= O(B),
\end{aligned}
$$

where the second equality follows from the fact that $\operatorname{cov}(\tilde{Z}_k^2 \tilde{Z}_i, \tilde{Z}_{k'}^2 \tilde{Z}_{j_1 j_2}) = 0$ for any $j_1 \neq j_2$ and $i, j_1, j_2 < k$; the third equality follows from the fact that $\operatorname{cov}(\tilde{Z}_k^2 \tilde{Z}_i, \tilde{Z}_{k'}^2 \tilde{Z}_j^2) = 0$ for any $i \neq j$ and $i, j < k$; and the last equality follows from the fact that $\sum_{t=1}^{k-1} a_{tk} a_{tk'}^2 \operatorname{cov}(\tilde{Z}_k^2 \tilde{Z}_t, \tilde{Z}_{k'}^2 \tilde{Z}_t^2)$ has at most B nonzero terms by the definition of a_k and $a_{k'}$ by Assumption 4 and similar arguments as in $\mathcal{T}_{11}$.

Similarly, the other cross-term $\mathcal{T}_{13}$ is also of order $O(B)$.

Next, consider $\mathcal{T}_{14}$ where we have

$$
\begin{aligned}
\mathcal{T}_{14} &= \operatorname{cov}\left\{\tilde{Z}_k^2 \left(\sum_{t=1}^{k-1} a_{tk}^2 \tilde{Z}_t^2 + 2 \sum_{1 \leq t_1 < t_2 \leq k-1} a_{t_1 k} a_{t_2 k} \tilde{Z}_{t_1} \tilde{Z}_{t_2}\right), \tilde{Z}_{k'}^2 \left(\sum_{t=1}^{k-1} a_{tk'}^2 \tilde{Z}_t^2 + 2 \sum_{1 \leq t_1 < t_2 \leq k-1} a_{t_1 k'} a_{t_2 k'} \tilde{Z}_{t_1} \tilde{Z}_{t_2}\right)\right\} \\
&= \operatorname{cov}\left(\tilde{Z}_k^2 \sum_{t=1}^{k-1} a_{tk}^2 \tilde{Z}_t^2, \tilde{Z}_{k'}^2 \sum_{t=1}^{k-1} a_{tk'}^2 \tilde{Z}_t^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\text{cov}\left(\tilde{Z}_k^2\sum_{t=1}^{k-1}a_{tk}^2\tilde{Z}_t^2, \tilde{Z}_{k'}^2\sum_{1\le t_1<t_2\le k-1}a_{t_1k'}a_{t_2k'}\tilde{Z}_{t_1}\tilde{Z}_{t_2}\right)\\
&+2\text{cov}\left(\tilde{Z}_k^2\sum_{1\le t_1<t_2\le k-1}a_{t_1k}a_{t_2k}\tilde{Z}_{t_1}\tilde{Z}_{t_2}, \tilde{Z}_{k'}^2\sum_{t=1}^{k-1}a_{tk'}^2\tilde{Z}_t^2\right)\\
&+4\text{cov}\left(\tilde{Z}_k^2\sum_{1\le t_1<t_2\le k-1}a_{t_1k}a_{t_2k}\tilde{Z}_{t_1}\tilde{Z}_{t_2}, \tilde{Z}_{k'}^2\sum_{1\le t_1<t_2\le k-1}a_{t_1k'}a_{t_2k'}\tilde{Z}_{t_1}\tilde{Z}_{t_2}\right)\\
= &\sum_{t=1}^{k-1}\text{cov}(\tilde{Z}_k^2a_{tk}^2\tilde{Z}_t^2, \tilde{Z}_{k'}^2a_{tk'}^2\tilde{Z}_t^2)+0+0+4\sum_{1\le t_1<t_2\le k-1}\text{cov}(\tilde{Z}_k^2a_{t_1k}a_{t_2k}\tilde{Z}_{t_1}\tilde{Z}_{t_2}, \tilde{Z}_{k'}^2a_{t_1k'}a_{t_2k'}\tilde{Z}_{t_1}\tilde{Z}_{t_2})\\
= &O(B)+O(kB^2) \;=\; O(kB^2),
\end{aligned}
$$

where the third equality follows from the facts that

1. $\text{cov}(\tilde{Z}_k^2\tilde{Z}_i^2, \tilde{Z}_{k'}^2\tilde{Z}_j^2)=0$ for any $i\neq j$ and $i,j<k$;
2. $\text{cov}(\tilde{Z}_k^2\tilde{Z}_i^2, \tilde{Z}_{k'}^2\tilde{Z}_{j_1}\tilde{Z}_{j_2})=0$ for any $j_1\neq j_2$ and $i,j_1,j_2<k$;
3. $\text{cov}(\tilde{Z}_k^2\tilde{Z}_{i_1}\tilde{Z}_{i_2}, \tilde{Z}_{k'}^2\tilde{Z}_j^2)=0$ for any $i_1\neq i_2$ and $i_1,i_2,j<k$; and
4. $\text{cov}(\tilde{Z}_k^2\tilde{Z}_{i_1}\tilde{Z}_{i_2}, \tilde{Z}_{k'}^2\tilde{Z}_{j_1}\tilde{Z}_{j_2})$ is nonzero if and only if $(i_1,i_2)=(j_1,j_2)$ for any $i_1,i_2,j_1,j_2<k$;

and the last equality follows from the facts that $\sum_{t=1}^{k-1}a_{tk}^2a_{tk'}^2\text{cov}(\tilde{Z}_k^2\tilde{Z}_t^2, \tilde{Z}_{k'}^2\tilde{Z}_t^2)$ has at most kB nonzero terms by Assumption 4 and similar arguments as in term $\mathcal{T}_{11}$ and $\sum_{1\le t_1<t_2\le k-1}a_{t_1k}a_{t_2k}a_{t_1k'}a_{t_2k'}\text{cov}(\tilde{Z}_k^2\tilde{Z}_{t_1}\tilde{Z}_{t_2}, \tilde{Z}_{k'}^2\tilde{Z}_{t_1}\tilde{Z}_{t_2})$ is nonzero only if t_1 and t_2 are both connected to both groups k and k'. Rearranging terms, we have

$$
\begin{aligned}
&\sum_{1\le t_1<t_2\le k-1}a_{t_1k}a_{t_2k}a_{t_1k'}a_{t_2k'}\text{cov}(\tilde{Z}_k^2\tilde{Z}_{t_1}\tilde{Z}_{t_2}, \tilde{Z}_{k'}^2\tilde{Z}_{t_1}\tilde{Z}_{t_2})\\
= &\underbrace{\sum_{t_1=1}^{k-1}a_{t_1k}a_{t_1k'}}_{\text{at most } B \text{ nonzero terms}}\left\{\underbrace{\sum_{t_2\neq t_1}a_{t_2k}a_{t_2k'}\text{cov}(\tilde{Z}_k^2\tilde{Z}_{t_1}\tilde{Z}_{t_2}, \tilde{Z}_{k'}^2\tilde{Z}_{t_1}\tilde{Z}_{t_2})}_{\text{at most } B \text{ nonzero terms}}\right\}\\
= &O(B^2).
\end{aligned}
$$

Combining the four terms $\mathcal{T}_{11}$–$\mathcal{T}_{14}$, we have $\mathcal{T}_1$ is of order $O(B^2)$, and thus $\text{cov}(\Delta_k^2,\Delta_{k'}^2)$ is also of order $O(B^2)$ because $\mathcal{T}_2$ and $\mathcal{T}_3$ are smaller order terms compared with $\mathcal{T}_1$.

S3.2 Positive semidefinite matrix

We prove several results regarding the properties of the matrices that appeared in Theorem 2.

Lemma S3 (Positive semi-definiteness of $p^{-[\Omega]}$ and $(1-p)^{-[\Omega]}$). *We have the following results:*

1. *The matrices $p^{-[\Omega]}$ and $(1-p)^{-[\Omega]}$ are both positive semidefinite matrices.*
2. *The following matrix is positive semidefinite:*

$$\begin{pmatrix} \Lambda_1 & \Lambda_\tau \\ \Lambda_\tau & \Lambda_0 \end{pmatrix}. \tag{S19}$$

Proof. 1. Recall the definition of $\Omega = (w_{ij})_{i,j\in[n]}$, where $w_{ij} = |\mathcal{S}_i \cap \mathcal{S}_j|$. Note the following binomial decomposition of the elements:

$$p^{-w_{ij}} = \Big(1 + \frac{1-p}{p}\Big)^{w_{ij}} = 1 + \frac{1-p}{p}\binom{w_{ij}}{1} + \Big(\frac{1-p}{p}\Big)^2\binom{w_{ij}}{2} + \cdots + \Big(\frac{1-p}{p}\Big)^{\bar{S}}\binom{w_{ij}}{\bar{S}}. \tag{S20}$$

Hence, we can express $p^{-[\Omega]}$ as

$$p^{-\Omega} = \Big(1 + \frac{1-p}{p}\Big)^{[\Omega]} = 1 + \frac{1-p}{p}\binom{[\Omega]}{1} + \Big(\frac{1-p}{p}\Big)^2\binom{[\Omega]}{2} + \cdots + \Big(\frac{1-p}{p}\Big)^{\bar{S}}\binom{[\Omega]}{\bar{S}}. \tag{S21}$$

Now we prove that each $\binom{[\Omega]}{k}$ is P.S.D.

(1) For $k=1$, $\binom{[\Omega]}{k} = \Omega$, and $|\mathcal{S}_i \cap \mathcal{S}_j|$ counts the number of overlapping groups that unit i and j belong to. Recall that $W(i, \dot{)}$ indicates the groups that unit i belongs to. This suggests w_{ij} is also given by $W(i,\cdot)W(j,\cdot)^{\mathrm{T}}$. In other words,

$$\binom{[\Omega]}{k} = WW^{\mathrm{T}} \succeq 0. \tag{S22}$$

(2) For $k=2$, $\binom{w_{ij}}{2}$ indicates the number of pairs of groups units i and j belong to. If we further define a new membership matrix $W_2 \in \mathbb{R}^{n\times m(m-1)/2}$, whose rows are indexed by units and columns by pairs of m groups, with entry $W_2(i,l)$ representing whether unit i belongs to the l-th pair. Then we can show that

$$\binom{[\Omega]}{2} = W_2 W_2^{\mathrm{T}} \succeq 0. \tag{S23}$$

Similarly, we can extend the above trick to $\binom{[\Omega]}{k}$. We define W_k as the membership matrix indicating whether unit i belongs to a k-tuple of the groups. Then we can show

$$\binom{[\Omega]}{k} = W_k W_k^{\mathrm{T}} \succeq 0. \tag{S24}$$

This concludes the claim.

2. Consider any

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^{2n}. \tag{S25}$$

We can compute that

$$\alpha^{\mathrm{T}} \left(p^{[-\Omega_1]} - 1 \right) \alpha = \left(\frac{1-p}{p} \right) \alpha^{\mathrm{T}} W_1 W_1^{\mathrm{T}} \alpha + \left(\frac{1-p}{p} \right)^2 \alpha^{\mathrm{T}} W_2 W_2^{\mathrm{T}} \alpha + \cdots \tag{S26}$$

Here,

$$\alpha^{\mathrm{T}} W_k = \left(\sum_{i \in \mathrm{tuple}(t)} \alpha_i \right)_{t \in \mathcal{T}_k}. \tag{S27}$$

Same for the $\beta^{\mathrm{T}} \left((1-p)^{-[\Omega_1]} - 1 \right) \beta$ part. Now

$$\alpha^{\mathrm{T}} \Omega_1 \beta = \sum \alpha_i \beta_j \mathbf{1}\{(\Omega_1)_{ij} \neq 0\}. \tag{S28}$$

Note that for any index set $\mathcal{J}$,

$$2 \left| \left(\sum_{i \in \mathcal{J}} \alpha_i \right) \left(\sum_{j \in \mathcal{J}} \beta_j \right) \right| = 2 \left| \sqrt{\left(\frac{1-p}{p} \right)^k} \left(\sum_{i \in \mathcal{J}} \alpha_i \right) \cdot \sqrt{\left(\frac{p}{1-p} \right)^k} \left(\sum_{j \in \mathcal{J}} \beta_j \right) \right| \tag{S29}$$

$$\leq \left(\frac{1-p}{p} \right)^k \left(\sum_{i \in \mathcal{J}} \alpha_i \right)^2 + \left(\frac{p}{1-p} \right)^k \left(\sum_{j \in \mathcal{J}} \beta_j \right)^2 \tag{S30}$$

Therefore, for i, j, $(\Omega_1)_{ij} \neq 0$, we do the following:

(1) in

$$\left(\frac{1-p}{p} \right) \alpha^{\mathrm{T}} W_1 W_1^{\mathrm{T}} \alpha + \left(\frac{p}{1-p} \right) \beta^{\mathrm{T}} W_1 W_1^{\mathrm{T}} \beta, \tag{S31}$$

we get $\binom{w_{ij}}{1}$ lower bound

$$-\binom{w_{ij}}{1} \alpha_i \beta_j; \tag{S32}$$

(2) in

$$\left(\frac{1-p}{p}\right)^2 \alpha^{\mathrm{T}} W_2 W_2^{\mathrm{T}} \alpha + \left(\frac{p}{1-p}\right)^2 \beta^{\mathrm{T}} W_2 W_2^{\mathrm{T}} \beta, \tag{S33}$$

we get $\binom{w_{ij}}{2}$ lower bound

$$\binom{w_{ij}}{2} \alpha_i \beta_j; \tag{S34}$$

(3) in

$$\left(\frac{1-p}{p}\right)^3 \alpha^{\mathrm{T}} W_3 W_3^{\mathrm{T}} \alpha + \left(\frac{p}{1-p}\right)^3 \beta^{\mathrm{T}} W_3 W_3^{\mathrm{T}} \beta, \tag{S35}$$

we get $\binom{w_{ij}}{2}$ lower bound

$$-\binom{w_{ij}}{3} \alpha_i \beta_j; \tag{S36}$$

This process can be conducted until $k = \bar{S}$. Hence, taking an addition, we can obtain

$$\Big\{ -\binom{w_{ij}}{1} + \binom{w_{ij}}{2} - \binom{w_{ij}}{3} + \dots \Big\} \alpha_i \beta_j = -\alpha_i \beta_j + (1-1)^{w_{ij}} \alpha_i \beta_j = -\alpha_i \beta_j. \tag{S37}$$

This concludes the proof.

□